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Short Distance Expansions of Correlation Functions in the Sine-Gordon Theory

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We examine the two-point correlation functions of the fields $\exp(i\alpha\Phi)$ in the sine-Gordon theory at all values of the coupling constant $\hat{\beta}$. Using conformal perturbation theory, we write down explicit integral expressions for every order of the short distance expansion. Using a novel technique analagous to dimensional regularisation, we evaluate these integrals for the first few orders finding expressions in terms of generalised hypergeometric functions. From these derived expressions, we examine the limiting forms at the points where the sine-Gordon theory maps onto a doubled Ising and the Gross-Neveu SU(2) models. In this way we recover the known expansions of the spin and disorder fields about criticality in the Ising model and the well known Kosterlitz-Thouless flows in the Gross-Neveu SU(2) model.

1 Introduction

The most important generally unsolved problem in the subject of massive integrable quantum field theory in two dimensions is the computation of correlation functions. A completely general approach to the study of the large distance expansion is well known: one can insert a multiparticle resolution of the identity between fields and thus obtain an infinite integral representation for the correlation function involving the form factors. Multiparticle form factors have been computed in a variety of models by Smirnov [2]. In principle, this form factor sum completely characterizes the correlation function. However the complexity of the form factors makes this representation difficult to utilize.

On the other hand, many quantum field theories have well defined short distance expansions. This short distance expansion is essentially independent of the large distance expansion discussed above; it is indeed extremely difficult to obtain one from the other. In this paper we develop the short distance expansion of sine-Gordon correlation functions. This is also an infinite integral representation, though the integrals can be much more compactly written than the large distance expansion involving the form factors. It is our aim to demonstrate that this short distance expansion is tractable.

The form of the two point correlation functions of the fields $\exp(\mathrm{i}\alpha\Phi)$ in the sine-Gordon model at the free-fermion point (SG_{ff}) is well understood. Because these fields in the SG_{ff} can be mapped onto a doubling of the spin/disorder fields in the Ising model (Ityzkson and Zuber [4]), the results found by Wu et al.[5] allow the expression of the correlators as solutions of a Painleve III non-linear differential equation. More recently, this result was extended in [7]. There, using a generalization of the techniques developed in [3], it was shown how correlators of the fields $\exp(\mathrm{i}\alpha\Phi)$, $0 < \alpha < 1$, are characterized by solutions to a non-linear equation, the sinh-Gordon equation. The proof of this result lay in expressing the large distance expansion of the correlators as Fredholm determinants through the use of form factors, and then deriving differential equations for the Fredholm determinants.

In this paper we present new results characterizing correlators of the above fields in conformal perturbation theory for values of the sine-Gordon coupling, $\hat{\beta}$, in the range $1 < \hat{\beta} < \sqrt{2}$. (Here, $\hat{\beta}$ is related to the conventional sine-Gordon coupling constant as $\hat{\beta} = \beta/\sqrt{4\pi}$, the free fermion point occuring at $\hat{\beta} = 1$.) Because of the U(1) charge symmetry of the conformal limit of

the sine-Gordon theory, we focus upon the following two correlators:

$$G(\alpha,-\alpha) = \langle \exp(i\alpha\Phi(z,\bar{z}))\exp(-i\alpha\Phi(0)); \qquad (1.1)$$
$$G(\hat{\beta}/2,\hat{\beta}/2) = \langle \exp(i\hat{\beta}\Phi(z,\bar{z})/2)\exp(i\hat{\beta}\Phi(0)/2). \qquad (1.2)$$

Expanding in powers of the mass parameter, λ, we are able to write integral expressions for every order of the short distance expansion. However, not all orders are finite for the given range of $\hat{\beta}$. For the correlator $G(\alpha,-\alpha)$, IR singularities are absent at $O(\lambda^{2n})$ given $\hat{\beta}^2 > 2 - 1/n$, while for the correlator $G(\hat{\beta}/2,\hat{\beta}/2)$, IR singularities do not appear at $O(\lambda^{2n+1})$ provided $\hat{\beta}^2 > (2n+1)/(n+1)$. UV singularites are absent at every order for both correlators provided $\hat{\beta} < \sqrt{2}$.

In principle it is possible to evaluate the integrals at every order. The singularities appear as poles in the resulting functions: at lower orders, gamma functions, at higher orders, generalised hypergeometric functions. However, the results at higher orders quickly become intractable and not particularly illuminating. Thus we restrict ourselves to evaluating the correlator, $G(\alpha,-\alpha)$, to $O(\lambda^2)$, and the correlator, $G(\hat{\beta}/2,\hat{\beta}/2)$, to $O(\lambda^3)$. Indeed, $G(\hat{\beta}/2,\hat{\beta}/2)$ at $O(\lambda^3)$ is already only finite for $\hat{\beta}^2 > 3/2$, one-half of the full range of interest, $1 < \hat{\beta}^2 < 2$.

The integral expressions in the perturbative expansion do not arise from a diagrammatic analysis. Thus it might seem necessary to explicitly subtract the contribution of the bubble diagrams. (Without such a subtraction, the terms in the expansion are formally divergent.) However, the technique, as first introduced by Dotsenko [9], we use to evaluate the terms renders them finite without explicit inclusion of the bubbles in a manner analagous to dimensional regularisation. Rather than analytically continuing the dimension of spacetime, we continue the parameter, $\hat{\beta}$. In this way not only are the integrals rendered finite, the bubble diagrams evaluate to zero.

At particular points in the coupling, the sine-Gordon theory is known to map onto familar theories. At $\hat{\beta} = 1$, sine-Gordon maps onto a doubled Ising model, and at $\hat{\beta} = \sqrt{2}$, it maps onto the SU(2) Gross-Neveu model. We examine our perturbative expansion at these points. At both of these points the perturbative series become infinite and renormalisation is required. At $\hat{\beta} = 1$ we introduce a novel renormalisation scheme to remove the IR divergences. At $\hat{\beta} = \sqrt{2}$ we employ a more conventional scheme to take care of the UV divergences. Having then performed the necessary renormalisation,

we obtain the known expansions of correlators for the order/disorder fields in the Ising model as well as the known Kosterlitz-Thouless flows in the Gross-Neveu SU(2) model. This provides some degree of confidence that our derived expressions away from the free fermion point are correct.

2 Overview of the Perturbative Expansion

The action for the sine-Gordon theory in Euclidean space-time (our conventions are $z = (t + ix)/2, \bar{z} = (t - ix)/2$ and $d^2z = -dtdx/2$) is

$$S = S_{CFT} + S_{Pert.} = -\frac{1}{4\pi}\int d^2z\Big(\partial_z\Phi\partial_{\bar{z}}\Phi + 4\lambda : cos(\hat{\beta}\Phi) :\Big) \tag{2.1}$$

where, again, the parameter $\hat{\beta}$ is related to the conventionally normalized coupling by $\hat{\beta} = \frac{\beta}{\sqrt{4\pi}}$. As shown by Coleman [8], the UV divergences in the theory for values of $\hat{\beta} < \sqrt{2}$ (i.e. for values of $\hat{\beta}$ where the coupling λ is not irrelavent), may be removed by the straight-forward normal ordering of the $\cos(\hat{\beta}\Phi)$. In the process, the coupling constant λ is multiplicatively renormalised. As no wavefunction renormalisation is necessary, the scaling dimensions of the fields remain unchanged by the introduction of the cosine perturbation - the dimensions are the same as those in the deep ultraviolet. Because the structure of the field theory is the same as its conformal limit, it is meaningful to perturb about the free theory - indeed, we may use the same labels for the fields in both the free and massive theories. For a general discussion of conformal perturbation theory see [1].

The coupling constant, λ, in 2.1 can be directly related to the mass, m, of the asymptotic states, the solitons. Zamolodchikov [11], using results derived with thermodynamic Bethe ansatz, demonstrated that

$$\lambda = -\tilde{\lambda}(\hat{\beta})m^{2-\hat{\beta}^2} \tag{2.2}$$

where [1]

$$\tilde{\lambda}(\hat{\beta}) = \frac{\Gamma(\hat{\beta}^2/2)}{\Gamma(1-\hat{\beta}^2/2)}\left[\frac{\sqrt{\pi}\Gamma(1/(2-\hat{\beta}^2)}{\Gamma(\hat{\beta}^2/(4-2\hat{\beta}^2)}\right]^{2-\hat{\beta}^2}. \tag{2.3}$$

[1] Some caution must be used in applying Zamolodchikov's result directly as he uses different conventions for both the action and the propagator.

We note that $\lambda = -m$ at $\hat{\beta} = 1$. This relation (2.2) will prove to be useful in our examination of the Ising model.

Consider the correlation functions

$$G(\alpha, \alpha') = \langle 0| : \exp(i\alpha\Phi(z, \bar{z})) :: \exp(i\alpha'\Phi(0)) : |0\rangle. \qquad (2.4)$$

These correlation functions are highly non-trivial. Even at the free-fermion point, $\hat{\beta} = 1$, the bosonisation relations, $: \exp(\pm\Phi) := \Psi_{\pm}\Psi_{\mp}$, where the Ψ's are the spinor components in the massive Thirring model, imply the fields, $\exp(i\alpha\Phi)$, are not simply representable as fermions.

The perturbative expansion of these correlators in a path integral formulation is

$$\begin{aligned} G(\alpha, \alpha') &= \frac{\int D\Phi e^{-S(\Phi)} e^{i\alpha\Phi(z)} e^{i\alpha'\Phi(0)}}{\int D\Phi e^{-S(\Phi)}} \qquad (2.5) \\ &= \frac{\int D\Phi e^{-S_{CFT}(\Phi)} e^{\frac{\lambda}{\pi}\int d^2w \ cos(\hat{\beta}\Phi(w))} e^{i\alpha\Phi(z)} e^{i\alpha'\Phi(0)}}{\int D\Phi e^{-S_{CFT}(\Phi)} e^{\frac{\lambda}{\pi}\int d^2w \ cos(\hat{\beta}\Phi(w))}} \\ &= \frac{\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\lambda}{\pi}\right)^n \int d^2w_i < cos(\hat{\beta}\Phi(w_1))\cdots cos(\hat{\beta}\Phi(w_n)) e^{i\alpha\Phi(z)} e^{i\alpha'\Phi(0)} >_{CFT}}{\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\lambda}{\pi}\right)^n \int d^2w_i < cos(\hat{\beta}\Phi(w_1))\cdots cos(\hat{\beta}\Phi(w_n)) >_{CFT}} \\ &= \frac{\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\lambda}{2\pi}\right)^n \int d^2w_i \sum_{l_k=\pm 1} < e^{il_1\hat{\beta}\Phi(w_1)}\cdots e^{il_n\hat{\beta}\Phi(w_n)} e^{i\alpha\Phi(z)} e^{i\alpha'\Phi(0)} >_{CFT}}{\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\lambda}{2\pi}\right)^n \int d^2w_i \sum_{l_k=\pm 1} < e^{il_1\hat{\beta}\Phi(w_1)}\cdots e^{il_n\hat{\beta}\Phi(w_n)} >_{CFT}} \end{aligned}$$

where we have expanded $e^{-S_{Pert}}$ in powers of λ and $<>_{CFT}$ indicates the correlator is to be evaluated in the conformal limit. The advantage of expressing $G(\alpha, \alpha')$ in this form is that the correlators of vertex operators in the conformal limit, i.e. $<>_{CFT}$, are easily evaluated. The denominators of these expressions, representing the bubble diagrams in the theory, will prove to be important in proving the convergence of the terms in the perturbative expansion.

To evaluate the resulting conformal correlators we need to specify the free boson propagator. With the action as given in 2.1 the propagator is

$$\langle\Phi(z, \bar{z})\Phi(0)\rangle = -\log(cz\bar{z}), \qquad (2.6)$$

where c is some arbitrary constant. As our convention we set $c = 1$. With this propagator the OPE of two vertex operators $: e^{i\alpha\Phi(z,\bar{z})} :: e^{i\beta\Phi(0)} :$, is then equal to

$$: e^{i\alpha\Phi(z,\bar{z})} :: e^{i\beta\Phi(0)} := \left(|z|^2\right)^{\alpha\beta} : e^{i(\alpha+\beta)\Phi(z,\bar{z})} : + \cdots. \tag{2.7}$$

Using this OPE, the terms of the form $< e^{i\alpha_1\Phi(z_1)} \cdots e^{i\alpha_n\Phi(z_n)} >_{CFT}$ are then given by

$$< e^{i\alpha_1\Phi(z_1)} \cdots e^{i\alpha_n\Phi(z_n)} >_{CFT} = \delta(\sum \alpha_i) \prod_{i<j} \left(|z_i - z_j|^2\right)^{\alpha_i\alpha_j}. \tag{2.8}$$

That $<>_{CFT}$ vanishes unless $\sum \alpha_i = 0$ is as consequence of the U(1) symmetry of the free massless theory.

Given this U(1) symmetry, only certain values of α and α' lead to non-vanishing correlators, $G(\alpha, \alpha')$, in the massive theory in perturbation theory. We examine two in particular, $G(\alpha, -\alpha)$ and $G(\hat{\beta}/2, \hat{\beta}/2)$. The first has non-vanishing terms at all even orders in the perturbation, the second at all odd orders. One reason for choosing to examine these correlators in particular is their relation to the Ising spin and disorder field correlators, $< \sigma(z, \bar{z})\sigma(0) >$ and $< \mu(z, \bar{z})\mu(0) >$, respectively. At $\hat{\beta} = 1$, these correlators are related to the G's via

$$< \sigma(z, \bar{z})\sigma(0) >^2 = < \sin(\frac{\Phi(z, \bar{z})}{2}) \sin(\frac{\Phi(0)}{2}) > = \frac{1}{2}\left(G(\frac{1}{2}, -\frac{1}{2}) - G(\frac{1}{2}, \frac{1}{2})\right) \tag{2.9}$$

and

$$< \mu(z, \bar{z})\mu(0) >^2 = < \cos(\frac{\Phi(z, \bar{z})}{2}) \cos(\frac{\Phi(0)}{2}) > = \frac{1}{2}\left(G(\frac{1}{2}, -\frac{1}{2}) + G(\frac{1}{2}, \frac{1}{2})\right), \tag{2.10}$$

where we have used $G(\alpha, \alpha') = G(-\alpha, -\alpha')$.

The perturbative expansion for $G(\alpha, -\alpha)$ and $G(\hat{\beta}/2, \hat{\beta}/2)$, using 2.8 and the U(1) symmetry, simplifies to

$$G(\alpha, -\alpha) = \frac{\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\lambda}{2\pi}\right)^{2n} \binom{2n}{n} \int d^2w_1 \cdots d^2w_{2n} \omega(\hat{\beta}, w_i)\tau(\alpha, \hat{\beta}, w_i)}{\sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{\lambda}{2\pi}\right)^{2n} \binom{2n}{n} \int d^2w_1 \cdots d^2w_{2n} \omega(\hat{\beta}, w_i)} \tag{2.11}$$

and

$$G(\frac{\hat{\beta}}{2},\frac{\hat{\beta}}{2}) = \frac{\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\lambda}{2\pi}\right)^{2n+1}\int d^2w_1\cdots d^2w_{2n+1}\binom{2n+1}{n+1}\chi(\hat{\beta},w_i)\psi(\hat{\beta},w_i)}{\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{\lambda}{2\pi}\right)^{2n}\binom{2n}{n}\int d^2w_1\cdots d^2w_{2n}\omega(\hat{\beta},w_i)} \tag{2.12}$$

where the functions ω, τ, χ, and ψ are given by

$$\omega(\hat{\beta},w_i) = \prod_{\substack{1\le i\neq j\le n\\ 2n\ge i\neq j>n}}(|w_i-w_j|^2)^{\hat{\beta}^2}\prod_{1\le i\le n<j\le 2n}(|w_i-w_j|^2)^{-\hat{\beta}^2}, \tag{2.13}$$

$$\tau(\alpha,\hat{\beta},w_i) = |z^{-2}|^{\alpha^2}\prod_{1\le i\le n}(|w_i-z|^2|w_i|^{-2})^{\alpha\hat{\beta}}\prod_{n<i\le 2n}(|w_i-z|^{-2}|w_i|^2)^{\alpha\hat{\beta}}, \tag{2.14}$$

$$\chi(\hat{\beta},w_i) = \prod_{\substack{1\le i\neq j\le n\\ 2n+1\ge i\neq j>n}}(|w_i-w_j|^2)^{\hat{\beta}^2}\prod_{1\le i\le n<j\le 2n+1}(|w_i-w_j|^2)^{-\hat{\beta}^2}, \tag{2.15}$$

$$\psi(\hat{\beta},w_i) = |z^2|^{\frac{\hat{\beta}}{4}}\prod_{1\le i\le n}(|w_i-z|^2|w_i|^2)^{\frac{\hat{\beta}^2}{2}}\prod_{n<i\le 2n+1}(|w_i-z|^2|w_i|^2)^{-\frac{\hat{\beta}^2}{2}}. \tag{2.16}$$

3 Convergence of Terms in Perturbative Expansion

In this section we examine the singularities that appear in the perturbative expansions of the correlators $G(\alpha,-\alpha)$ and $G(\hat{\beta}/2,\hat{\beta}/2)$. We find that $G(\alpha,-\alpha)$ at $O(\lambda^{2n})$ is UV finite for $\hat{\beta}^2<2$ and IR finite for $\hat{\beta}^2>2-1/n$, and that $G(\hat{\beta}/2,\hat{\beta}/2)$ at $O(\lambda^{2n+1})$ is again UV finite for $\hat{\beta}^2<2$ but IR finite for $\hat{\beta}^2>(2n+1)/(n+1)$. We give a proof of this beginning with $G(\alpha,-\alpha)$.

To show UV finiteness, we consider the contribution to $G(\alpha,-\alpha)$ at $O(\lambda^{2n})$. It takes the form

$$\int d^2w_1\cdots d^2w_{2n}\langle e^{i\alpha\Phi(z)}e^{-i\alpha\Phi(0)}e^{i\hat{\beta}\Phi(w_1)}e^{-i\hat{\beta}\Phi(w_2)}\cdots e^{i\hat{\beta}\Phi(w_{2n-1})}e^{-i\hat{\beta}\Phi(w_{2n})}\rangle \\ -\text{ disconnected pieces.} \tag{3.1}$$

The leading order UV singularity occurs as $w_1 \to w_2$, $w_3 \to w_4$, ..., $w_{2n-1} \to w_{2n}$. These singularities are governed by the operator product expansion

$$
\begin{aligned}
: e^{i\hat{\beta}\Phi(x)} :: e^{-i\hat{\beta}\Phi(y)} : &= |x-y|^{-2\hat{\beta}^2} : e^{i\hat{\beta}(\Phi(x)-\Phi(y))} : + \text{ finite pieces} \qquad (3.2)\\
&= |x-y|^{-2\hat{\beta}^2} \left(I - \hat{\beta}^2 |x-y|^2 : \partial_y \Phi(y) \partial_{\bar{y}} \Phi(y) : + \cdots \right),
\end{aligned}
$$

where in the second line pieces such as $(x-y)\partial_y\Phi(y)$ and its antiholomorphic counterpart are discarded. Such pieces, when the OPE is inserted into the correlator, integrate out.

The leading order singularity in this term does not come from the contribution of the identity operator in the above OPE; rather this term leads to a disconnected piece which is subsequently subtracted off. Thus the leading order singularity that contributes is $|x-y|^{2-2\hat{\beta}^2}$. So the leading UV singularity at $O(\lambda^{2n})$ in $G(\alpha, -\alpha)$ has dimension

$$
d_{UV} = n(2 - 2\hat{\beta}^2 + 2). \qquad (3.3)
$$

The n arises as there n approaches $w_i \to w_{i+1}$, the extra 2 from the integration $\int d^2(w_i - w_{i+1})$. Hence for $\hat{\beta}^2 < 2$, i.e. $d_{UV} > 0$, the terms are UV finite, as is expected from Coleman [8].

The reader may worry that stronger divergences are introduced considering the approaches $w_1 \to w_3$, $w_5 \to w_7$, and so forth in addition to $w_1 \to w_2$, $w_3 \to w_4$, etc. But such additional singularities are governed by the OPE

$$
\begin{aligned}
&: \partial_x \Phi(x) \partial_{\bar{x}} \Phi(x) :: \partial_y \Phi(y) \partial_{\bar{y}} \Phi(y) := \qquad (3.4)\\
&\qquad |x-y|^{-4} \left(I + |x-y|^2 : \partial_y \Phi(y) \partial_{\bar{y}} \Phi(y) + \cdots \right).
\end{aligned}
$$

Again the leading term in the OPE is disconnected. So its leading singular contribution is $|x-y|^{-2}$. But this is exactly cancelled by the integration $\int d^2(x-y)$ and so the overall UV singularity becomes no greater.

When all the w_i's are allowed to approach one another, successively substituting the above OPEs into the correlator in 3.1 leads one eventually to the correlator

$$
\langle e^{i\alpha\Phi(z)} e^{-\alpha\Phi(0)} : \partial_w \Phi(w) \partial_{\bar{w}} \Phi(\bar{w}) :\rangle. \qquad (3.5)
$$

But this correlator, as can be easily checked, introduces no additional UV singularities as w approaches z or 0. Thus allowing in addition the w_i's

to approach z or 0 does not change the UV properties of the perturbative expansion.

To demonstrate that $G(\alpha, -\alpha)$ is IR finite at $O(\lambda^{2n})$ for $\hat{\beta}^2 > 2 - 1/n$, we recast each term in the perturbative expansion by making the conformal transformation

$$w_i \to 1/w_i. \tag{3.6}$$

The perturbative terms then appear as

$$\begin{aligned}\int |w_1|^{2\hat{\beta}^2-4} d^2 w_1 \cdots |w_{2n}|^{2\hat{\beta}^2-4} d^2 w_{2n} \times \\ \langle e^{i\alpha\Phi(z)} e^{-i\alpha\Phi(0)} e^{i\hat{\beta}\Phi(w_1)} e^{-i\hat{\beta}\Phi(w_2)} \cdots e^{i\hat{\beta}\Phi(w_{2n-1})} e^{-i\hat{\beta}\Phi(w_{2n})} \rangle \\ - \text{disconnected pieces.}\end{aligned} \tag{3.7}$$

The IR behaviour of this integral is determined by sending $w_i \to 0$ for all i. The singular behaviour of the correlator in the above integral can be extracted by first sending $w_1 \to w_2$, $w_3 \to w_4$, $\ldots$, $w_{2n-1} \to w_{2n}$, then $w_1 \to w_3$, $w_5 \to w_7$, $\ldots$, $w_{2n-3} \to w_{2n-1}$, and so on. Using the OPEs given in 3.2 and 3.4 and discarding, as before, the disconnected pieces associated with the identity, the dimension of the leading IR singularity is

$$d_{IR} = \left[2n\left(2\hat{\beta}^2 - 2\right)\right] + \left[2 - 2n\hat{\beta}^2\right]. \tag{3.8}$$

The latter term in d_{IR} is the contribution coming from the correlator, the former term from $\int |w_i|^{2\hat{\beta}^2-4} d^2 w_i$. It is important to not include the singularities that arise from the OPEs involving $e^{i\alpha\Phi(0)}$. These singularities are properly UV. Then for the term to be IR finite, we need $d_{IR} > 0$. Thus $\hat{\beta}^2$ must be greater than $2 - 1/n$.

To check the UV finiteness of $G(\hat{\beta}/2, \hat{\beta}/2)$, a similiar argument is used to the one above. At $O(\lambda^{2n+1})$ the term in the perturbative expansion appears as

$$\begin{aligned}\int d^2 w_1 \cdots d^2 w_{2n+1} \times \\ \langle e^{i\hat{\beta}/2\Phi(z)} e^{i\hat{\beta}/2\Phi(0)} e^{-i\hat{\beta}\Phi(w_1)} e^{i\hat{\beta}\Phi(w_2)} e^{-i\hat{\beta}\Phi(w_3)} \cdots e^{i\hat{\beta}\Phi(w_{2n})} e^{-i\hat{\beta}\Phi(w_{2n+1})} \rangle \\ - \text{disconnected pieces.}\end{aligned} \tag{3.9}$$

The most singular pieces of this expression come when $w_2 \to w_3$, $w_4 \to w_5$, $\ldots$, $w_{2n} \to w_{2n+1}$. These approaches are governed by the OPE in 3.2. Thus

the overall dimension of the leading UV singularity is

$$d_{UV} = n(4 - 2\hat{\beta}^2), \tag{3.10}$$

and so again for $\hat{\beta}^2 < 2$ the terms are UV finite. Additional approaches, say $w_2 \to w_4$, $w_6 \to w_8$, etc., are govered by the OPE 3.4 and so, as before, do not change d_{UV}. If all the w_i's, $i > 1$, are allowed to approach one another, say at w, the succession of OPEs leaves one to evaluate the correlator

$$\langle e^{i\hat{\beta}/2\Phi(z)} e^{i\hat{\beta}/2\Phi(0)} e^{-i\beta\Phi(w_1)} : \partial_w \Phi(w) \partial_{\bar{w}} \Phi(w) :\rangle. \tag{3.11}$$

But as $w \to w_1$, 0, or z, no new UV singularities appear.

To obtain the IR behaviour of $G(\hat{\beta}/2, \hat{\beta}/2)$ for $n > 0$, we begin by treating the term at $O(\lambda^{2n+1})$ like the term at $O(\lambda^{2n})$ in $G(\alpha, -\alpha)$, contracting the last 2n vertex operators $e^{\pm i\hat{\beta}\Phi(w)}$ in the correlator. However there is an additional contribution of $2\hat{\beta}^2 - 4$ to d_{IR} in this case that comes from letting $w \to w_1$ in 3.11. Thus the term is IR finite if $\hat{\beta}^2 > (2n+1)/(n+1)$. For $n = 0$ it is easy to see that this formula still holds: the term at $O(\lambda)$ is IR finite if $\hat{\beta}^2 > 1$.

Though we have identified the ranges of $\hat{\beta}$ where the perturbative expansion is divergent, we point out that the expressions we derive in the next section will only be divergent for individual values of $\hat{\beta}$: in general the expressions we will derive will be meromorphic functions of $\hat{\beta}$. However it is not clear how to interpret the analytically continued expressions in regions of $\hat{\beta}$ where the original integral expressions are divergent. It is possible that that this continuation does not capture all the relavent physics. For example at $\hat{\beta}^2 < 1$, it may be the continued correlators do not reflect the presence of quantum breathers.

4 Evaluation of Terms of Perturbative Series

We begin by focusing on the integrals in the numerator of the perturbative expansion (i.e. we ignore the contribution of the bubble diagrams). Although these integrals are formally divergent, the techniques (as first developed by Dotsenko [9]) we employ to evaluate them render them finite in a manner somewhat analogous to dimensional regularisation. This handling of the infinities suggests the technique takes into account the bubble diagrams. This

is born out by the fact that when the contribution of the bubbles are calculated explicitly using the same techniques, they turn out to be identically zero.

The integrals we thus consider are of the form

$$I_{2n} = \int d^2w_1 \cdots d^2w_{2n}\omega(\hat{\beta}, w_1, \cdots, w_{2n})\tau(\alpha, \hat{\beta}, w_1, \cdots, w_{2n}), \tag{4.1}$$

and

$$I_{2n+1} = \int d^2w_1 \cdots d^2w_{2n+1}\chi(\hat{\beta}, w_1, \cdots, w_{2n+1})\psi(\hat{\beta}, w_1, \cdots, w_{2n+1}). \tag{4.2}$$

These integrals are, in their full generality, unwieldly. Thus, in the end, we only explicitly evaluate I_1, I_2, and I_3. Indeed, I_3 is already only finite for $\hat{\beta}^2 > 3/2$, a part of the full range of interest. However, we will carry out the calculation for general n as far as seems reasonable. We begin with I_{2n}.

4.1 Evaluation of I_{2n}

To make sense of I_{2n} we must first abandon light cone coordinates, writing it in the usual x-t coordinates. Doing so, and at the same time scaling out the z-dependence, we obtain

$$\begin{aligned} I_{2n} &= (-2)^{2n}|z|^{4n-2\alpha^2+2\hat{\beta}^2 n}\int dt_1 dx_1 \cdots dt_{2n}dx_{2n} \times \\ &\prod_{\substack{1\leq i\neq j\leq n \\ n<i\neq j\leq 2n}}\Big((t_i-t_j)^2+(x_i-x_j)^2\Big)^{\hat{\beta}} \prod_{1\leq i\leq n<j\leq 2n}\Big((t_i-t_j)^2+(x_i-x_j)^2\Big)^{-\hat{\beta}} \times \\ &\prod_{1\leq i\leq n}\left(\Big((t_i-1)^2+x_i^2\Big)^2\Big(t_i^2+x_i^2\Big)^{-2}\right)^{\alpha\hat{\beta}} \times \\ &\prod_{n<i\leq 2n}\left(\Big((t_i-1)^2+x_i^2\Big)^{-2}\Big(t_i^2+x_i^2\Big)^{2}\right)^{\alpha\hat{\beta}}. \end{aligned} \tag{4.3}$$

Now the branch points in I_{2n}'s integrand of the x_i's lie on the imaginary axis. Thus the deformation of the x-contours pictured in Figure 1 is permissible. This deformation corresponds to the following change of variables $x_i \to -ie^{-i2\epsilon}x_i$ where ϵ is some small positive parameter. Given that the

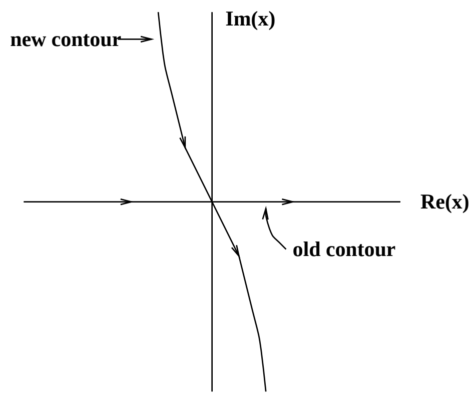

Figure 1: A sketch of the deformation of the x_i contours.

terms in the integral now take the form $(t_i - t_j)^2 - e^{-i4\epsilon}(x_i - x_j)^2$, the following changes of variables

$$w_i^{\pm} = t_i \pm x_i$$

factorises these expressions leaving I_{2n} in the form

$$I_{2n} = (-2)^{2n} \left(\frac{i}{2}\right)^{2n} |z|^{4n-2\alpha^2+2\hat{\beta}^2 n} \times \tag{4.4}$$
$$\int dw_1^+ \cdots dw_{2n}^+ dw_1^- \cdots dw_{2n}^- \theta(w_1^+, \cdots, w_{2n}^+, \epsilon)\theta(w_1^-, \cdots, w_{2n}^-, -\epsilon)$$

where $\theta(w_1, \cdots, w_{2n}, \epsilon)$ is given by

$$\theta(w_1, \cdots, w_{2n}, \epsilon) \quad = \quad \left[\prod_{i=1}^{n} (w_i - 1 - i\epsilon\Delta_i)^{\alpha\hat{\beta}} (w_i - i\epsilon\Delta_i)^{-\alpha\hat{\beta}} \times \right. \tag{4.5}$$
$$\left. \prod_{j=i+1}^{n} (w_i - w_j - i\epsilon(\Delta_i - \Delta_j))^{\hat{\beta}^2} \prod_{j=n+1}^{2n} (w_i - w_j - i\epsilon(\Delta_i - \Delta_j))^{-\hat{\beta}^2} \right] \times$$
$$\left[\prod_{i=n+1}^{2n} (w_i - 1 - i\epsilon\Delta_i)^{-\alpha\hat{\beta}} (w_i - i\epsilon\Delta_i)^{\alpha\hat{\beta}} \prod_{j=i+1}^{2n} (w_i - w_j - i\epsilon(\Delta_i - \Delta_j))^{\hat{\beta}^2}\right],$$

and $\Delta_i = w_i^+ - w_i^-$.

Having made this set of changes of variable, we are faced with a set of integrals that are represented by contours deformed around the branch points of algebraic functions. The signs of the coefficients of ϵ in the integrand tell us

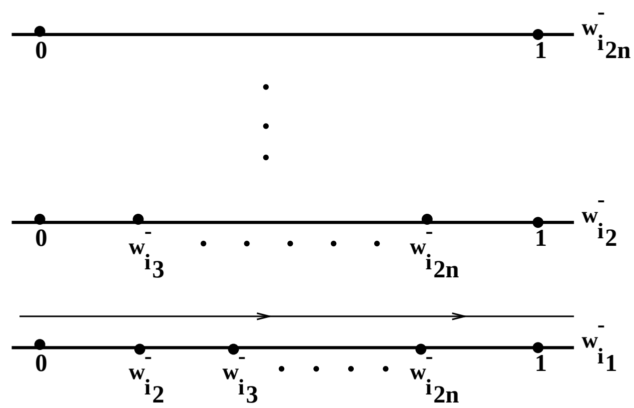

Figure 2: The marked points on each of the contours represent the branch points. We have not drawn the contours for $w^-_{i_2}, \cdots, w^-_{i_{2n}}$ as they are irrelavent to the argument being made.

how to make the deformations. Focusing on the w_i^--contours, the coefficients of the relavent ϵ's are functions of the variables w_i^+. For various values of the w_i^+'s, the w_i^--contours will enclose no poles, and the corresponding contribution to I_{2n} is 0. To find these values we consider a set w_i^+ ordered as follows:

$$w^+_{i_1} > w^+_{i_2} > \cdots > w^+_{i_{2n}}.$$

We will show that if $w^+_{i_1} > 1$ or $w^+_{i_{2n}} < 0$, the contribution to I_{2n} is zero.

First suppose $w^+_{i_1} > 1$. Then if we perform the w_i^- integrations in ascending order of i, the set of contours for w_i^- appear as in Figure 2. Because the $w^-_{i_1}$-contour can be closed at ∞, the contribution to I_{2n} is 0. Only the branch points of w_i^- arising from the terms of the form $(w_i^- - a + i\epsilon f(w_k^+, w_k^-))^\gamma$ are shown. The branch points of w_i^- arising from terms of the form $(w_i^+ - a + i\epsilon f(w_k^+, w_k^-))^\gamma$, i.e. terms where w_i^- appears multiplied by ϵ, are along the imaginary axis of w_i^- and are taken to $\pm i\infty$ as $\epsilon \to 0$. Thus they do not affect the deformation of the contour.

Now suppose $w^+_{i_n} < 0$. The set of contours for w_i^+ then appear as in Figure 3. Because the w^-_{2n} contour can be closed in the lower half plane, the contribution to I_{2n} is again 0.

In addition to the branch points that arise directly from the terms $(w_i^- - a + i\epsilon f(w_k^+, w_k^-))$, branch points will also appear because of prior integrations. For example, after performing the first 2n-1 w^--integrations, I_{2n} is left in the

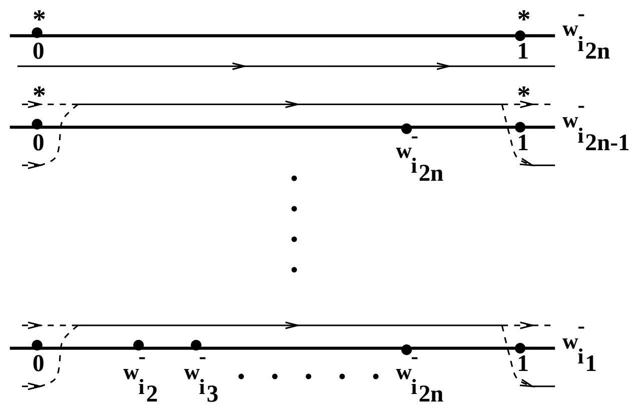

Figure 3: Pictured are the set of contours for w_i^- when $w_{2n}^+ < 0$. The dashed lines indicate the contours may follow either path, depending on the specific values of the w_i^-'s. The $*$'s mark branch points arising from prior integrations, as explained in the text.

form

$$I_{2n} = \int dw_1^+ \cdots dw_{2n}^+ dw_{2n}^- \theta(w_1^+, \cdots, w_{2n}^+, \epsilon) \times \left(w_{2n}^- - 1 + i\epsilon\Delta_{2n}\right)^{-\alpha\hat{\beta}} \left(w_{2n}^- + i\epsilon\Delta_{2n}\right)^{\alpha\hat{\beta}} g(w_{2n}^- - i\epsilon), \tag{4.6}$$

where g is a generalised hypergeometric function with branch points at 0 and 1. These branch points arise from integrating terms of the form $\left(w_k^- - w_{2n}^- + i\epsilon(\Delta_k - \Delta_{i_{2n}})\right)^\gamma$, $k \neq i_{2n}$. As the coefficent of ϵ in this term is positive when $w_k^- = w_{i_{2n}}^-$ (as $w_k^+ - w_{i_{2n}}^+ > 0$), g depends on $w_{2n}^- - i\epsilon$ as indicated. Hence the $w_{i_{2n}}$-contour flows underneath the branch points of g. The importance of all of this is, of course, that g's branch points do not interfere with the deformation of the contour that takes the contribution to 0.

Thus is has been shown that only for values of w_i^+ constrained to lie between 0 and 1 is a non-zero contribution to I_{2n} made. As the are n! ways of ordering the w_i^+'s, there are n! cases to consider, each having the general form

$$1 > w_{i_1}^+ > \cdots > w_{i_{2n}}^+ > 0.$$

The contours for w_i^-'s in this general case appear as in Figure 4. The branch points marked by the *'s now appear below the contours as we are performing

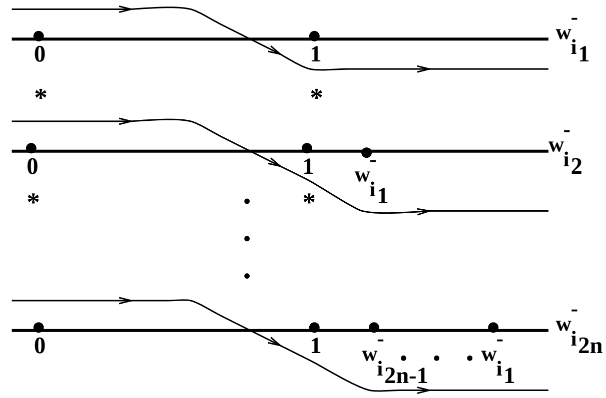

Figure 4: Pictured are the positions of the set of contours for w_i^- relative to the branch points when the w_i^+ are between 0 and 1.

the w_{i_j} integrals in descending order of j. The placement of the $w_{i_j}^-$ branch points beyond 1 is justified by the deformation of the contours that we now make as shown in Figure 5. As we see, the contours are now such that the values of the w_i^-'s are all greater than 1, thus justifying the placement of the w_i^- branch points beyond 1. However, the given ordering of these branch points should not be taken as fixed, but should be understood to depend upon the specific values of the w_i^-'s. For example, if we are at a point on the contour C_2 such that $w_{i_2}^- > w_{i_1}^-$ then for the contours C_k, $k > 2$ the branch point for $w_{i_1}^-$ should always fall to the right of that of $w_{i_2}^-$. Thus, for example, the ordering of the branch points for the contour C_{2n} as pictured in Figure 5 assumes an integration region for the other 2n-1 contours where $w_{i_{2n-1}}^- < \cdots < w_{i_1}^-$.

Now taking $\epsilon \to 0$, we thus are able to express I_{2n} as the following sum of integrals

$$I_{2n} = (-i)^{2n} |z|^{4n-2\alpha^2-2\hat{\beta}^2 n} \sum_{\sigma \in S_{2n}} I\left(\sigma(1), \cdots, \sigma(2n)\right), \tag{4.7}$$

where S_{2n} is the permutation group of 2n objects and

$$I\left(i_1, \cdots, i_{2n}\right) = J_1\left(i_1, \cdots, i_{2n}\right) J_2\left(i_1, \cdots, i_{2n}\right), \tag{4.8}$$

where J_1 and J_2 are given by

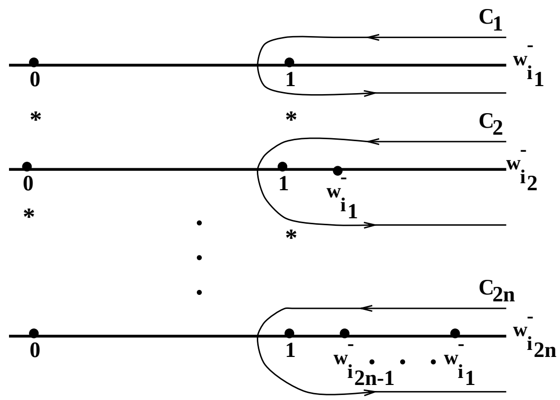

Figure 5: Pictured are the deformations of the contours employed when w_1^+ through w_n^+ are between 0 and 1.

$$
\begin{aligned}
J_1(i_1,\cdots,i_{2n}) \;=\; & \int_0^1 dw_{i_1} w_{i_1}^{-\gamma_{i_1}}(1-w_{i_1})^{\gamma_{i_1}} \times \\
& \int_0^{w_{i_1}} dw_{i_2} w_{i_2}^{-\gamma_{i_2}}(1-w_{i_2})^{\gamma_{i_2}}(w_{i_1}-w_{i_2})^{\gamma_{i_1 i_2}} \int_0^{w_{i_3}} dw_{i_3}\cdots\times \\
& \int_0^{w_{2n-1}} dw_{i_{2n}} w_{i_{2n}}^{-\gamma_{i_{2n}}}(1-w_{i_{2n}})^{\gamma_{i_{2n}}}(w_{i_1}-w_{i_{2n}})^{\gamma_{i_1 i_{2n}}}\cdots \\
& \qquad (w_{i_{2n-1}}-w_{i_{2n}})^{\gamma_{i_{2n-1} i_{2n}}}
\end{aligned} \tag{4.9}
$$

$$
\begin{aligned}
J_2(i_1,\cdots,i_{2n}) \;=\; & \int_{C_1} dw_{i_1} w_{i_1}^{-\gamma_{i_1}}(1-w_{i_1})^{\gamma_{i_1}} \times \\
& \int_{C_2} dw_{i_2} w_{i_2}^{-\gamma_{i_2}}(1-w_{i_2})^{\gamma_{i_2}}(w_{i_1}-w_{i_2})^{\gamma_{i_1 i_2}} \int_{C_3} dw_{i_3}\cdots\times \\
& \int_{C_{2n}} dw_{i_{2n}} w_{i_{2n}}^{-\gamma_{i_{2n}}}(1-w_{i_{2n}})^{\gamma_{i_{2n}}}(w_{i_1}-w_{i_{2n}})^{\gamma_{i_1 i_{2n}}}\cdots\times \\
& \qquad (w_{i_{2n-1}}-w_{i_{2n}})^{\gamma_{i_{2n-1} i_{2n}}}
\end{aligned} \tag{4.10}
$$

and we have defined the γ's as follows

$$
\gamma_i = \begin{cases} \alpha\hat{\beta}/2 & 1 \le i \le n \\ -\alpha\hat{\beta}/2 & n < i \le 2n \end{cases}; \tag{4.11}
$$

$$
\gamma_{ij} = \begin{cases} \hat{\beta}^2 & 1 \le i,j \le n \text{ or } n < i,j \le 2n \\ -\hat{\beta}^2 & {\scriptstyle 1\le i\le n \atop n<j\le 2n} \text{ or } {\scriptstyle 1\le j\le n \atop n<i\le 2n} \end{cases}. \tag{4.12}
$$

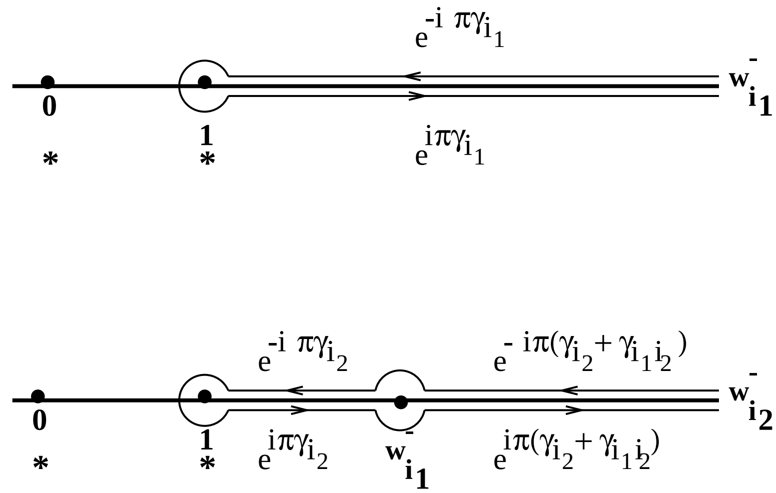

Figure 6: Shown are the phase assignments for the integral $J_2(i_1, i_2)$.

In summing over the $J_1 J_2$'s we rearrange the signs of the terms involving powers of $w_i - w_j$. But because we always do this in pairs, one for J_1 and one for J_2, there are no phases between the terms in the sum $\sum_{\sigma \in S_n}$. We note that $I(i_1, \ldots, i_{2n})$ is invariant under various permutations of the i_j's. Any permutation not mixing values of the i_j's between 1 through n with those n+1 through 2n leaves $I(i_1, \ldots, i_{2n})$ unchanged.

Now J_1 is no more than a generalised Euler integral which we evaluate in Appendix A. The form is sufficiently complicated for the general case that there is little point in writing it down here. Indeed, 4.9 offers a much compact expression for J_1. J_2, on the other hand, is as of yet still a formal expression. The contours C_k enclose multiple branch points and the corresponding cuts introduce phases. These phases depend upon which section of the integration region we are in, as explained earlier. These sections can be labelled by the specific order of the w_i^-'s in the section. As there are n! ways of ordering the w_i^-'s, J_2 is a sum of n! terms. Now it is possible to develop a general notation for the assignment of the phases in each of the n! cases. However, this does not bring us any farther in actually evaluating I_{2n}. Thus it is at this point that we specialize to specific values of n, in particular $n = 1$. This case, together with the evaluation of I_3, brings out all the subtleties inherit in the assignment of phases.

4.2 Evaluation of I_2

Assigning phases is particularly straightforward for the case of I_2. From Figure 5, n=1 yields phases as pictured in Figure 6. $J_2(i_1,i_2)$ then has the following form:

$$\begin{aligned} J_2(i_1,i_2) = \quad & 2is(\gamma_{i_1})\int_1^\infty dw_{i_1}(w_{i_1}-1)^{\gamma_{i_1}} w_{i_1}^{-\gamma_{i_1}} \times \\ & \Big[2is(\gamma_{i_2})\int_1^{w_{i_1}} dw_{i_2}(w_{i_2}-1)^{\gamma_{i_2}} w_{i_2}^{-\gamma_{i_2}}(w_{i_1}-w_{i_2})^{\gamma_{i_1 i_2}} + \\ & 2is(\gamma_{i_2}+\gamma_{i_1 i_2})\int_{w_{i_1}}^\infty dw_{i_2}(w_{i_2}-1)^{\gamma_{i_2}} w_{i_2}^{-\gamma_{i_2}}(w_{i_1}-w_{i_2})^{\gamma_{i_1 i_2}}\Big] \end{aligned} \tag{4.13}$$

where we have introduced the notation $s(a)=\sin(\pi a)$. In writing this expression for $J_2(i_1,i_2)$ down we have assumed that the portions of the contours that circle about the branch point contribute nothing. This assumption can always be made good by continuing the exponents of the terms in the integrands to points where these portions do contribute nothing, evaluating the integral, and then continuing back to the desired value of the exponents in the final expression. It is this continuation of the exponents that makes the method akin to dimensional regularisation. But instead of continuing the dimension of space-time, we continue the the parameter $\hat\beta$, i.e. the scaling dimension of the vertex operators appearing in the action.

$J_2(i_1,i_2)$ is now in a form where it may be directly evaluated. Making the change of variables, $w_i \to 1/w_i$, the above may be written as

$$J_2(i_1,i_2) = -4\left[s(\gamma_{i_1})s(\gamma_{i_2})K(i_2,i_1) + s(\gamma_{i_1})s(\gamma_{i_2}+\gamma_{i_1 i_2})K(i_1,i_2)\right], \tag{4.14}$$

where $K(i,j)$ is defined to be

$$K(i,j) = \int_0^1 dw_i w_i^{-2+\hat\beta^2}(1-w_i)^{\gamma_i}\int_0^{w_i} dw_j w_j^{-2+\hat\beta^2}(1-w_j)^{\gamma_j}(w_i-w_j)^{-\hat\beta^2}, \tag{4.15}$$

where we have used the fact that $\gamma_{12}=\hat\beta^2$ and that $\gamma_1=-\gamma_2$. We evaluate $K(i,j)$ in Appendix B (even though the K's are no more than Euler integrals, Appendix A is not sufficient to evaluate the K's) with the result

$$\begin{aligned} K(i_1,i_2) \quad = \quad & \mathrm{B}(1+\gamma_{i_1},\hat\beta^2-1)\Gamma(1-\hat\beta^2)\Gamma(\hat\beta^2-1)(-\gamma_{i_2})(\hat\beta^2-1) \\ & \times {}_3F_2(1-\gamma_{i_2},\hat\beta^2-1,\hat\beta^2,\hat\beta^2+\gamma_{i_1},2,1), \end{aligned} \tag{4.16}$$

where ${}_3F_2$ is a standard hypergeometric function.

Using Appendix A to directly evaluate $J_1(i_1, i_2)$ we find

$$\begin{aligned}
J_1(i_1,i_2) &= \int_0^1 dw_{i_1} w_{i_1}^{-\gamma_{i_1}}(1-w_{i_1})^{\gamma_{i_1}} \times \\
&\qquad \int_0^{w_{i_1}} dw_{i_2} w_{i_2}^{-\gamma_{i_2}}(1-w_{i_2})^{\gamma_{i_2}}(w_{i_1}-w_{i_2})^{\gamma_{12}} \qquad (4.17)\\
&= B(1+\gamma_{i_1}, 2-\hat{\beta}^2)B(1-\hat{\beta}^2, 1-\gamma_{i_2}) \times \\
&\qquad {}_3F_2(-\gamma_{i_2}, 2-\hat{\beta}^2, 1-\gamma_{i_2}, 3-\hat{\beta}^2+\gamma_{i_1}, 2-\hat{\beta}^2-\gamma_{i_2}, 1).
\end{aligned}$$

So I_2 has the form

$$\begin{aligned}
I_2 = -4|z|^{4-2\alpha^2-2\hat{\beta}^2} \Big[& J_1(1,2)\left(s^2(\tfrac{\alpha\hat{\beta}}{2})K(2,1) + s(\tfrac{\alpha\hat{\beta}}{2})s(\hat{\beta}^2+\tfrac{\alpha\hat{\beta}}{2})K(1,2)\right) \\
& + J_1(2,1)\left(s^2(\tfrac{\alpha\hat{\beta}}{2})K(1,2) + s(\tfrac{\alpha\hat{\beta}}{2})s(\tfrac{\alpha\hat{\beta}}{2}-\hat{\beta}^2)K(2,1)\right)\Big].
\end{aligned} \qquad (4.18)$$

4.3 Evaluation of I_{2n+1}

We now move on the I_{2n+1}. All of the same techniques that were used to analyze I_{2n} apply here. We are thus able to immediately write I_{2n+1} as follows:

$$I_{2n+1} = (-i)^{2n+1}|z|^{2(2n+1)-\hat{\beta}^2(2n+\frac{3}{2})} \sum_{\sigma\in S_{2n+1}} I\left(\sigma(1),\cdots,\sigma(2n+1)\right), \qquad (4.19)$$

where $I = (i_1,\cdots,i_{2n+1}) = J_1\left(i_1,\cdots,i_{2n+1}\right) J_2\left(i_1,\cdots,i_{2n+1}\right)$, and J_1 and J_2 are given by

$$\begin{aligned}
J_1\left(i_1,\cdots,i_{2n+1}\right) &= \int_0^1 dw_{i_1} w_{i_1}^{\delta_{i_1}}(1-w_{i_1})^{\delta_{i_1}} \times \qquad (4.20)\\
&\quad \int_0^{w_{i_1}} dw_{i_2} w_{i_2}^{\delta_{i_2}}(1-w_{i_2})^{\delta_{i_2}}(w_{i_1}-w_{i_2})^{\delta_{i_1 i_2}} \int_0^{w_{i_3}} dw_{i_3}\cdots \times \\
&\quad \int_0^{w_{2n}} dw_{i_{2n+1}} w_{i_{2n+1}}^{\delta_{i_{2n+1}}}(1-w_{i_{2n+1}})^{\delta_{i_{2n+1}}}(w_{i_1}-w_{i_{2n+1}})^{\delta_{i_1 i_{2n+1}}} \\
&\quad \cdots (w_{i_{2n}}-w_{i_{2n+1}})^{\delta_{i_{2n} i_{2n+1}}}
\end{aligned}$$

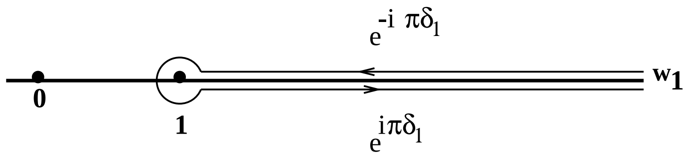

Figure 7: Shown are the phase assignments for the integral $J_2(1)$.

$$
\begin{aligned}
J_2\left(i_1, \cdots, i_{2n}\right) &= \int_{C_1} dw_{i_1} w_{i_1}^{\delta_{i_1}} (1-w_{i_1})^{\delta_{i_1}} \times \\
&\quad \int_{C_2} dw_{i_2} w_{i_2}^{\delta_{i_2}} (1-w_{i_2})^{\delta_{i_2}} (w_{i_1}-w_{i_2})^{\delta_{i_1 i_2}} \int_{C_3} dw_{i_3} \cdots \times \\
&\quad \int_{C_{2n+1}} dw_{i_{2n+1}} w_{i_{2n+1}}^{\delta_{i_{2n+1}}} (1-w_{i_{2n}})^{\delta_{i_{2n+1}}} (w_{i_1}-w_{i_{2n+1}})^{\delta_{i_1 i_{2n+1}}} \\
&\qquad\qquad \cdots (w_{i_{2n}}-w_{i_{2n+1}})^{\delta_{i_{2n} i_{2n+1}}}
\end{aligned} \tag{4.21}
$$

and we have defined the δ's as follows

$$
\delta_i = \begin{cases} -\hat{\beta}^2/2 & 1 \leq i \leq n+1 \\ \hat{\beta}^2 & n+1 < i \leq 2n+1 \end{cases} \tag{4.22}
$$

and

$$
\delta_{ij} = \begin{cases} \hat{\beta}^2 & 1 \leq i \neq j \leq n+1 \text{ or } n+1 < i \neq j \leq 2n+1 \\ -\hat{\beta}^2 & \substack{1 \leq i \leq n+1 \\ n+1<j\leq 2n+1} \text{ or } \substack{1 \leq j \leq n+1 \\ n+1<i\leq 2n+1} \end{cases} \tag{4.23}
$$

Again J_2 is a formal object because as of yet phases have not yet been assigned to the various portions of the contours C_k. And again there is no advantage of performing this assignment in general and so we specialize to the cases of I_1 and I_3.

4.4 Evaluation of I_1 and I_3

Now I_1 is particularly simple. The phase assignment is given in Figure 7. Thus $J_2(1)$ equals

$$
\begin{aligned}
J_2(1) &= 2is(\delta_1) \int_1^\infty dw_1 w_1^{\delta_1} (w_1-1)^{\delta_1} \\
&= 2is(\delta_1) \int_0^1 dw_1 w_1^{-2-2\delta_1} (1-w_1)^{\delta_1}.
\end{aligned} \tag{4.24}
$$

Now $J_1(1)$ and $J_2(1)$ are no more than the integral representations of the beta-function. Thus

$$
\begin{aligned}
J_1(1) &= \mathrm{B}(1+\delta_1, 1+\delta_1), &(4.25)\\
J_2(1) &= 2is(\delta_1)\mathrm{B}(-1-2\delta_1, 1+\delta_1), &(4.26)
\end{aligned}
$$

and so because $\delta_1 = -\hat{\beta}^2/2$

$$
I_1 = -2|z|^{2-3\hat{\beta}^2/2} s(\hat{\beta}^2/2)\mathrm{B}(1-\frac{\hat{\beta}^2}{2}, 1-\frac{\hat{\beta}^2}{2})\mathrm{B}(\hat{\beta}^2-1, 1-\frac{\hat{\beta}^2}{2}). \tag{4.27}
$$

We mention that the integral I_1 is one that also arises in calculating the 4-point Virasoro-Sharpiro formula. The methods used in this case for evaluating this integral give the same answer as above.

The evaluation of I_3 is more complicated because the assignment of phases is more complicated. The assignment differs depending upon the ordering of w_{i_1} with w_{i_2}. The phase assignment for the two possible orderings is pictured in Figure 8. Defining $K(i_1, i_2, i_3)$ by

$$
\begin{aligned}
K(i_1, i_2, i_3) &= \int_1^\infty dw_{i_1} {w_{i_1}}^{\delta_{i_1}} (w_{i_1}-1)^{\delta_{i_1}} \times &(4.28)\\
&\quad \int_1^{w_{i_1}} dw_{i_2} {w_{i_2}}^{\delta_{i_2}} (w_{i_2}-1)^{\delta_{i_2}} (w_{i_1}-w_{i_2})^{\delta_{i_1}\delta_{i_2}} \times \\
&\quad \int_1^{w_{i_2}} dw_{i_3} {w_{i_3}}^{\delta_{i_3}} (w_{i_3}-1)^{\delta_{i_3}} (w_{i_1}-w_{i_3})^{\delta_{i_1 i_3}} (w_{i_2}-w_{i_3})^{\delta_{i_2 i_3}},
\end{aligned}
$$

which after a change of variables $w_i \to 1/w_i$ can be written as

$$
\begin{aligned}
K(i_1, i_2, i_3) &= \int_0^1 dw_{i_3} {w_{i_3}}^{-2-2\delta_{i_3}-\delta_{i_3 i_2}-\delta_{i_3 i_1}} (1-w_{i_3})^{\delta_{i_3}} &(4.29)\\
&\quad \int_0^{w_{i_3}} dw_{i_2} {w_{i_2}}^{-2-2\delta_{i_2}-\delta_{i_2 i_3}-\delta_{i_2 i_1}} (1-w_{i_2})^{\delta_{i_2}} (w_{i_3}-w_{i_2})^{\delta_{i_3 i_2}} \\
&\quad \int_0^{w_{i_2}} dw_{i_1} {w_{i_1}}^{-2-2\delta_{i_1}-\delta_{i_1 i_3}-\delta_{i_1 i_2}} (1-w_{i_1})^{\delta_{i_1}} (w_{i_3}-w_{i_1})^{\delta_{i_3 i_1}} \times \\
&\qquad (w_{i_2}-w_{i_1})^{\delta_{i_1 i_2}},
\end{aligned}
$$

allows us to write $J_2(i_1, i_2, i_3)$ as

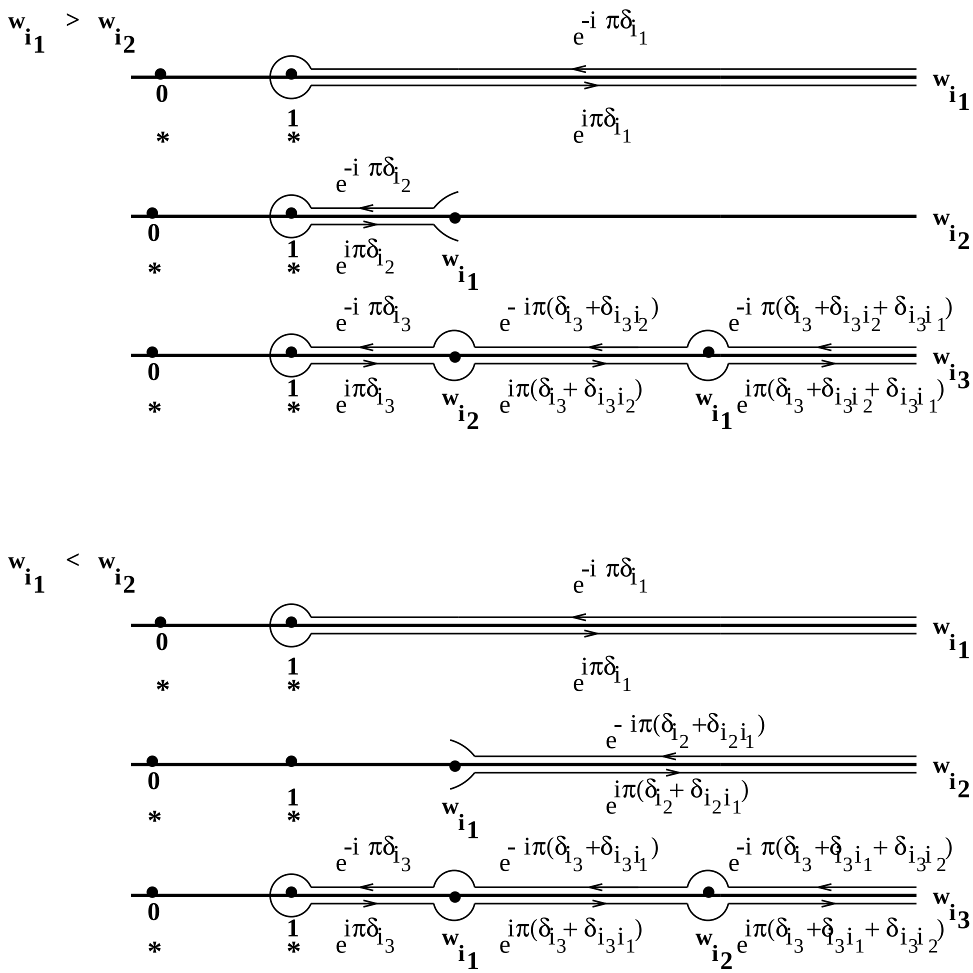

Figure 8: Shown are the phase assignments for the integral $J_2(i_1, i_2, i_3)$.

$$
\begin{aligned}
J_2(i_1,i_2,i_3) = (2i)^3 \quad [& s(\delta_{i_1})s(\delta_{i_2})s(\delta_{i_3})K(i_1,i_2,i_3)+ \\
& s(\delta_{i_1})s(\delta_{i_2})s(\delta_{i_3}+\delta_{i_3 i_2})K(i_1,i_3,i_2)+ \\
& s(\delta_{i_1})s(\delta_{i_2})s(\delta_{i_3}+\delta_{i_3 i_2}+\delta_{i_3 i_1})K(i_3,i_1,i_2)+ \\
& s(\delta_{i_1})s(\delta_{i_2}+\delta_{i_1 i_2})s(\delta_{i_3})K(i_2,i_1,i_3)+ \\
& s(\delta_{i_1})s(\delta_{i_2}+\delta_{i_1 i_2})s(\delta_{i_3}+\delta_{i_3 i_1})K(i_2,i_3,i_1)+ \\
& s(\delta_{i_1})s(\delta_{i_2}+\delta_{i_1 i_2})s(\delta_{i_3}+\delta_{i_3 i_1}+\delta_{i_3 i_2})K(i_3,i_2,i_1)]\,.
\end{aligned} \tag{4.30}
$$

Because $K_(i_1,i_2,i_3)$ is invariant under permutations of the i's involving 1 and 2, it can easily be shown from the above expression that

$$
\begin{aligned}
J_2(1,2,3) &= 0; & (4.31)\\
J_2(1,3,2) &= (2i)^3 s^2(\hat\beta^2/2)(s(3\hat\beta^2/2)+s(\hat\beta^2/2))K(1,2,3); & (4.32)\\
J_2(3,1,2) &= (2i)^3 s(\hat\beta^2/2)s(3\hat\beta^2/2)(s(3\hat\beta^2/2)+s(\hat\beta^2/2))K(1,2,3)+ & \\
&\quad s^2(\hat\beta^2/2)(s(\hat\beta^2/2)+s(3\hat\beta^2/2))K(1,3,2). & (4.33)
\end{aligned}
$$

$J_2(1,2,3)$ vanishes indentically because of a cancellation of the phases. Using the fact that both J_1 and J_2 possess the same symmetry as K, I_3 then takes the form

$$
\begin{aligned}
I_3 = 16|z|^{6-7\hat\beta^2/2} \quad \Big[& K(1,2,3)\left(s^2(\hat\beta^2/2)(s(3\hat\beta^2/2)+s(\hat\beta^2/2))J_1(1,3,2)+\right. \\
& \left. s(\hat\beta^2/2)s(3\hat\beta^2/2)(s(3\hat\beta^2/2)+s(\hat\beta^2/2))J_1(3,1,2)\right)+ \\
& K(1,3,2)\left(s^2(\hat\beta^2/2)(s(3\hat\beta^2/2)+s(\hat\beta^2/2))J_1(3,1,2)\right)\Big]\,.
\end{aligned} \tag{4.34}
$$

Now all that remains is to evaluate the J_1's and the K's.

Using Appendix A we find

$$
\begin{aligned}
K(1,2,3) \;=\; & \mathrm{B}(1+\hat\beta^2/2,-3+2\hat\beta^2)\mathrm{B}(1-\hat\beta^2,-2+3\hat\beta^2)\mathrm{B}(1+\hat\beta^2,\hat\beta^2-1) \\
& \times\sum_{k_1k_2k_3}^{\infty}\frac{(\hat\beta^2/2)_{k_1}(\hat\beta^2/2)_{k_2}(\hat\beta^2)_{k_3}}{k_1!k_2!k_3!}\frac{(-3+2\hat\beta^2)_{k_1+k_2}}{(-2+5\hat\beta^2/2)_{k_1+k_2}} \\
& \times\frac{(-2+3\hat\beta^2)_{k_1+k_2+k_3}}{(-1+2\hat\beta^2)_{k_1+k_2+k_3}}\frac{(-1+\hat\beta^2)_{k_2+k_3}}{(2\hat\beta^2)_{k_2+k_3}};
\end{aligned} \tag{4.35}
$$

$$
\begin{aligned}
K(1,3,2) &= \mathrm{B}(1-\hat\beta^2/2,-3+2\hat\beta^2)\Gamma(1-\hat\beta^2)\Gamma(-2+\hat\beta^2)\Gamma(1-\hat\beta^2)\Gamma(\hat\beta^2-1) \\
&\quad\times\sum_{k_1k_2k_3}^{\infty}\frac{(-\hat\beta^2/2)_{k_1}(\hat\beta^2/2)_{k_2}(-\hat\beta^2)_{k_3}}{k_1!k_2!k_3!}\frac{(-3+2\hat\beta^2)_{k_1+k_2}}{(-2+3\hat\beta^2/2)_{k_1+k_2}} \\
&\qquad\times\frac{(-2+\hat\beta^2)_{k_1+k_2+k_3}}{\Gamma(-1+k_1+k_2+k_3)}\frac{(-1+\hat\beta^2)_{k_2+k_3}}{\Gamma(k_2+k_3)}; \qquad (4.36) \\
J_1(1,3,2) &= \mathrm{B}(1-\hat\beta^2/2,3-3\hat\beta^2/2)\mathrm{B}(1-\hat\beta^2,2-\hat\beta^2)\mathrm{B}(1-\hat\beta^2,1-\hat\beta^2/2) \\
&\quad\times\sum_{k_1k_2k_3}^{\infty}\frac{(-\hat\beta^2/2)_{k_1}(\hat\beta^2/2)_{k_2}(-\hat\beta^2)_{k_3}}{k_1!k_2!k_3!}\frac{(3-3\hat\beta^2/2)_{k_1+k_2}}{(4-2\hat\beta^2)_{k_1+k_2}} \\
&\qquad\times\frac{(2-\hat\beta^2)_{k_1+k_2+k_3}}{(3-2\hat\beta^2)_{k_1+k_2+k_3}}\frac{(1-\hat\beta^2/2)_{k_2+k_3}}{(2-3\hat\beta^2/2)_{k_2+k_3}}; \qquad (4.37) \\
J_1(3,1,2) &= \mathrm{B}(1+\hat\beta^2/2,3-3\hat\beta^2/2)\mathrm{B}(1-\hat\beta^2,2)\mathrm{B}(1+\hat\beta^2,1-\hat\beta^2/2) \\
&\quad\times\sum_{k_1k_2k_3}^{\infty}\frac{(\hat\beta^2/2)_{k_1}(\hat\beta^2/2)_{k_2}(\hat\beta^2)_{k_3}}{k_1!k_2!k_3!}\frac{(3-3\hat\beta^2/2)_{k_1+k_2}}{(4-\hat\beta^2)_{k_1+k_2}} \\
&\qquad\times\frac{(2)_{k_1+k_2+k_3}}{(3-\hat\beta^2)_{k_1+k_2+k_3}}\frac{(1-\hat\beta^2/2)_{k_2+k_3}}{(2+\hat\beta^2/2)_{k_2+k_3}}. \qquad (4.38)
\end{aligned}
$$

In evaluating these integrals the same sort of techniques that went into evaluating $K(i_1,i_2)$ (i.e. handling the appearance of a pole multiplying a zero - see Appendix B) were employed. This completes the evaluation of I_3.

4.5 Evaluation of the Bubbles

We now examine the contribution of the bubble diagrams (i.e. the integrals in the denominator of the perturbative expansion). We will show that with our techniques for evaluating the integrals, the bubbles vanish identically. The integrals that we must evaluate are

$$
I^B_{2n} = \int d^2w_1\cdots d^2w_{2n}\prod_{\substack{1\le i\neq j\le n\\ 2n\ge i\neq j>n}}(|w_i-w_j|^2)^{\hat\beta^2}\prod_{1\le i\le n<j\le 2n}(|w_i-w_j|^2)^{-\hat\beta^2}. \quad (4.39)
$$

Performing the change of variables $x_i \to -ie^{-i2\epsilon}x_i$, $w_i^{\pm}=t_i\pm x_i$ as before, I^B_{2n} becomes

$$
I^B_{2n} \;=\; (-i)^{2n}\int dw_i^+dw_i^-\eta(w_1^+,\cdots,w_{2n}^+,\epsilon)\eta(w_1^-,\cdots,w_{2n}^-,-\epsilon),
$$

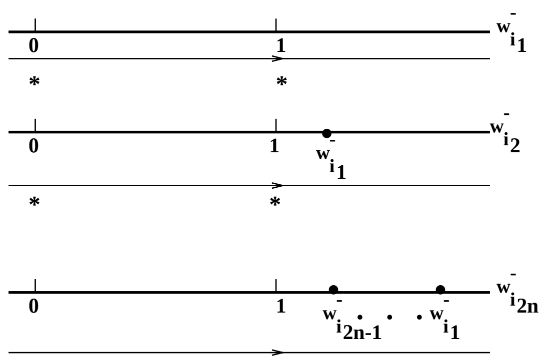

Figure 9: Pictured are the set of contours for $w^-_{i_1}$ arising from the integration of the integral I^B_{2n}. In contrast to the situation for I_{2n} and I_{2n+1}, there are no branch points at 0 and 1.

where η takes the from

$$\eta(w_1, \cdots, w_{2n}, \epsilon) = \prod_{i=1}^{2n} \prod_{j=i+1}^{n} (w_i - w_j - i\epsilon(\Delta_i - \Delta_j))^{\pm\hat{\beta}^2}, \tag{4.40}$$

where the sign of the exponents deponds upon the particular values of i and j. Again taking the ordering of the w^+_i's to be $w^+_{i_1} > \cdots w^+_{i_{2n}}$, the deformation of the w^-_i contours about the branch points the appear as in Figure 9.

Because there are no branch points at 0 and 1, the $w^-_{i_1}$-contour (or, indeed, the $w^-_{i_{2n}}$-contour) can always be closed at ∞. Thus I^B_{2n} is identically 0. In Figure 9 a specific ordering of the w^-_i branch points has been assumed. However, changing this ordering does not change the argument just made. The vanishing of the bubbles is supported by our analysis of the Ising spin correlators. Because our results match previous calculations, it is likely that we have accurately taken into account the perturbative structure of the vacuum.

4.6 Summary of Results

At this point we gather previous results to provide expressions for $G(\alpha,-\alpha)$ and $G(\hat{\beta}/2,\hat{\beta}/2)$:

$$\begin{aligned}
G(\alpha,-\alpha) &= |z|^{-2\alpha^2}+\left(\frac{\lambda}{2\pi}\right)^2 I_2 \qquad (4.41)\\
&= |z|^{-2\alpha^2}-\left(\frac{\lambda}{\pi}\right)^2 |z|^{4-2\alpha^2-2\hat{\beta}^2}\times\\
&\qquad \Big[J_1(1,2)\left(s^2(\tfrac{\alpha\hat{\beta}}{2})K(2,1)+s(\tfrac{\alpha\hat{\beta}}{2})s(\hat{\beta}^2+\tfrac{\alpha\hat{\beta}}{2})K(1,2)\right)+\\
&\qquad J_1(2,1)\left(s^2(\tfrac{\alpha\hat{\beta}}{2})K(1,2)+s(\tfrac{\alpha\hat{\beta}}{2})s(\tfrac{\alpha\hat{\beta}}{2}-\hat{\beta}^2)K(2,1)\right)\Big];
\end{aligned}$$

$$\begin{aligned}
G(\frac{\hat{\beta}}{2},\frac{\hat{\beta}}{2}) &= \left(\frac{\lambda}{2\pi}\right) I_1+\frac{1}{2}\left(\frac{\lambda}{2\pi}\right)^3 I_3 \qquad (4.42)\\
&= -\frac{\lambda}{\pi}|z|^{2-3\hat{\beta}^2/2}s(\hat{\beta}^2/2)\mathrm{B}(1-\hat{\beta}^2/2,1-\hat{\beta}^2/2)\mathrm{B}(\hat{\beta}^2-1,1-\hat{\beta}^2/2)\\
&\quad +\left(\frac{\lambda}{\pi}\right)^3 |z|^{6-7\hat{\beta}^2/2}\times\\
&\qquad \Big[K(1,2,3)\left(s^2(\hat{\beta}^2/2)(s(3\hat{\beta}^2/2)+s(\hat{\beta}^2/2))J_1(1,3,2)+\right.\\
&\qquad \left.s(\hat{\beta}^2/2)s(3\hat{\beta}^2/2)(s(3\hat{\beta}^2/2)+s(\hat{\beta}^2/2))J_1(3,1,2)\right)+\\
&\qquad K(1,3,2)\left(s^2(\hat{\beta}^2/2)(s(3\hat{\beta}^2/2)+s(\hat{\beta}^2/2))J_1(3,1,2)\right)\Big].
\end{aligned}$$

The expressions for $J(i,j)$ and $K(i,j)$ are given in 4.16 and 4.17 respectively. The expressions for $J(i,j,k)$ and $K(i,j,k)$ are given in 4.35 through 4.38.

5 Sine-Gordon as Ising and SU(2) Gross-Neveu

In this section we examine our expressions in the limits in which the sine-Gordon theory maps onto a doubled Ising model ($\hat{\beta}=1$) and the Gross-Neveu SU(2) model ($\hat{\beta}=\sqrt{2}$). We first examine the doubled Ising model.

5.1 Doubled Ising Model

As demonstrated in section 3, as $\hat{\beta}$ approaches 1, the perturbative expansion develops IR singularities. In our expressions, these singularities manifest themselves as poles in gamma functions. These poles are indicative of the logs that appear in the Ising spin-correlators in the scaling limit (see, for example, [5]). Specifically, a n-th order pole term indicates a n-th order log. The logs are obtained by expanding the powers of $|z|$ multiplying the pole terms. For example, writing $\hat{\beta}^2 = 1 + \epsilon$ with ϵ small, a typical situation will see the following expansion:

$$\frac{|z|^{1-\hat{\beta}^2}}{\hat{\beta}^2 - 1} = \frac{1}{\epsilon} - \log|z| + \cdots \ . \tag{5.1}$$

After this expansion is made, the divergent pieces (i.e. the $1/\epsilon$ term) are subtracted away. This can be understood in analogy with a cut-off method of regularisation. Throwing away the divergent terms amounts to no more than a specific choice of the cutoff. This procedure differs from that used by Dotsenko [9] where the powers of $|z|$ are not expanded and logs are substituted directly for the $1/\epsilon$ terms. This procedure is equivalent to ours in situations involving first order logs. However it fails to produce the correct coefficients of higher order logs that arise from overlapping divergences.

To test this regularisation technique, we examine a correlator simpler than the Ising correlator, the free-fermion correlator. As such we will need to review some aspects of the connection between the free-fermion and the sine-Gordon theory at $\hat{\beta} = 1$. The free fermion action in Euclidean space-time reads

$$S = \frac{1}{4\pi} \int dx dt \left[\bar{\psi}_- \partial_z \bar{\psi}_+ + \psi_- \partial_{\bar{z}} \psi_+ + im(\psi_- \bar{\psi}_+ - \bar{\psi}_- \psi_+) \right], \tag{5.2}$$

where we have introduced the Dirac spinors $\Psi_\pm = \binom{\bar{\psi}_\pm}{\psi_\pm}$ which satisfy $\Psi_+ = \Psi_-^\dagger$ and employed an appropriate choice of gamma matrices [2]. The equations of motion for these fields are

$$\begin{aligned} \partial_z \bar{\psi}_\pm &= im\psi_\pm, \\ \partial_{\bar{z}} \psi_\pm &= -im\bar{\psi}_\pm. \end{aligned} \tag{5.3}$$

[2] $\gamma^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\gamma^1 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$.

The correlators we will examine are

$$\langle\bar{\psi}_-(z,\bar{z})\psi_+(0)\rangle \;\; \text{and} \;\; \langle\psi_-(\mathrm{z},\bar{\mathrm{z}})\psi_+(0)\rangle. \tag{5.4}$$

Knowing the first of these correlators, the equations of motion allow us to fix the second. With this in mind, the on-shell mode expansions are:

$$\begin{aligned}\psi_\pm &= \pm\sqrt{m}\int_{-\infty}^{\infty}\frac{d\theta}{2\pi i}e^{\theta/2}\left(c^\mp(\theta)e^{\mp ip(\theta)\cdot x} - d^\pm(\theta)e^{\pm ip(\theta)\cdot x}\right)\\ \bar{\psi}_\pm &= -i\sqrt{m}\int_{-\infty}^{\infty}\frac{d\theta}{2\pi i}e^{-\theta/2}\left(c^\mp(\theta)e^{\mp ip(\theta)\cdot x} + d^\pm(\theta)e^{\pm ip(\theta)\cdot x}\right).\end{aligned} \tag{5.5}$$

In writing these expansions we have introduce the rapidity variable θ by which on-shell Euclidean momentum-energy is parameterized,

$$E = im\cosh(\theta) \;\; \text{and} \;\; \mathrm{p} = \mathrm{m}\sinh(\theta), \tag{5.6}$$

so that $-E^2 - p^2 = m^2$ and $ip(\theta)\cdot x = m(ze^\theta + \bar{z}e^{-\theta})$. The creation (+) and destruction (-) operators obey the following commutation relations

$$\{d^-(\theta), d^+(\theta')\} = \{c^-(\theta), c^+(\theta')\} = 4\pi^2\delta(\theta-\theta'). \tag{5.7}$$

Using these mode expansions together with the equations of motion, it is then straightforward to calculate the two correlators

$$\langle\bar{\psi}_-(z,\bar{z})\psi_+(0)\rangle = -2im\mathrm{K}_0(\mathrm{mr}), \tag{5.8}$$

$$\langle\psi_-(z,\bar{z})\psi_+(0)\rangle = 2m\sqrt{\frac{\bar{z}}{z}}\mathrm{K}_1(\mathrm{mr}). \tag{5.9}$$

where the K's are the standard modified Bessel functions.

The relavence of these correlators is found in the demonstration of Mandlestam [10] that the spinor components can be expressed in terms of the quasi-chiral components of the sine-Gordon field, ϕ^L and ϕ^R:

$$\begin{aligned}\psi_\pm &= \exp(\pm i\phi^L),\\ \bar{\psi}_\pm &= \exp(\mp i\phi^R),\end{aligned} \tag{5.10}$$

The fields ϕ^L and ϕ^R are termed quasi-chiral because in the massless limit,

$$\partial_{\bar{z}}\phi^L = \partial_z\phi^R = 0. \tag{5.11}$$

In this limit ϕ^L is the left mover and ϕ^R the right mover.

With these conventions we can make the following identification

$$\langle e^{i\phi^R(\bar{z})}e^{i\phi^L(0)}\rangle = -2im\mathrm{K}_0(\mathrm{mr}). \tag{5.12}$$

At first order the l.h.s. of this equation reduces to

$$\begin{aligned} F(\hat{\beta}=1) &\equiv \frac{\lambda}{2\pi}\int d^2w\langle e^{i\phi^R(\bar{z})}e^{i\phi^L(0)}e^{-i\Phi(w,\bar{w})}\rangle_{CFT} \\ &= -i\frac{\lambda}{2\pi}\int d^2w\frac{1}{(\bar{w}-\bar{z})}\frac{1}{w}. \end{aligned} \tag{5.13}$$

The i arises from taking apart $e^{i\Phi}$, i.e. $e^{\Phi(w,\bar{w})} = -ie^{-i\phi^L(w)}e^{-i\phi^R(\bar{w})}$.

This integral has a logarithmic divergence. To regulate it we continue the scaling dimensions of the operators (via continuing $\hat{\beta}$) as follows:

$$\begin{aligned} F(\hat{\beta}) &= \frac{\lambda}{2\pi}\int d^2w\langle e^{i\hat{\beta}\phi^R(\bar{z})}e^{i\hat{\beta}\phi^L(0)}e^{-i\hat{\beta}\Phi(w,\bar{w})}\rangle \\ &= -i\frac{\lambda}{2\pi}\int d^2w\frac{1}{(\bar{w}-\bar{z})^{\hat{\beta}^2}}\frac{1}{w^{\hat{\beta}^2}}. \end{aligned} \tag{5.14}$$

Notice that the U(1) charge of the vertex operators still sums to zero. Writing this integral in terms of Euclidean x-t coordinates, we easily evaluate it to be

$$F(\hat{\beta}) = \frac{2i\lambda}{\pi}|z|^{2-2\hat{\beta}^2}\frac{s(2-2\hat{\beta}^2)}{2-2\hat{\beta}^2}\mathrm{B}(2\hat{\beta}^2-2, 2-\hat{\beta}^2). \tag{5.15}$$

We now intend to take $\hat{\beta}^2$ to 1 via $\hat{\beta}^2 = 1+\epsilon$. λ then has scaling dimensions of $1-\epsilon$. So we may express λ as $\lambda = -m\mu^{-\epsilon}$ where μ is a mass scale fixed to be

$$\mu^{-\epsilon} = (1-\epsilon\gamma + O(\epsilon^2))m^{-\epsilon} \tag{5.16}$$

by Zamolodchikov's $\lambda - m$ relationship (see 2.2). Taking $\epsilon \to 0$ in $F(\hat{\beta})$ then gives us

$$F(1+\epsilon) = -\frac{i}{\epsilon}m\mu^{\epsilon}|z\mu|^{-2\epsilon} \tag{5.17}$$

$$= -im\mu^{\epsilon}\left(\frac{1}{\epsilon} - 2\log(m|z|) - 2\gamma\right) \tag{5.18}$$

As indicated before we drop the $1/\epsilon$ term. The above expression then reduces to

$$F(1) = 2im\left(\log(\frac{mr}{2}) + \gamma\right) \tag{5.19}$$

as $|z| = r/2$. In taking the limit $\epsilon \to 0$ we have not expanded out the μ^ϵ term so as to preserve the scaling dimension of $F(\hat{\beta})$. Expanding out $\mathrm{K}_0(\mathrm{mr})$ to first order in m shows the two sides of 5.12 to be in agreement. We thus have gained some confidence that our methods of evaluating the perturbative terms are correct.

To illustrate the connection between our method of regularisation and a cut-off method of regularisation, considering regulating $F(1)$ with a cut-off $|w| < R$:

$$F(1) = -i\frac{\lambda}{2\pi}\int^R d^2w \frac{1}{(\bar{w} - \bar{z})}\frac{1}{w}. \tag{5.20}$$

Doing the integral we find

$$F(1) = 2i\lambda\log(\frac{R}{|z|}) + const. \tag{5.21}$$

Now R can be written as $R = abm^{-1}$ where a and b are dimensionless constants and a is to be taken to ∞ (as the cutoff is removed). So as $\lambda = -m$

$$F(1) = 2im\log(a^{-1}b^{-1}m|z|) + const. \tag{5.22}$$

To obtain a match between $F(1)$ and the known form of $\langle\bar{\psi}_-(z,\bar{z})\psi_+(0)\rangle$ $\log(a)$ must be thrown away and a specific choice for b must be made. The throwing away of $\log(a)$ corresponds to our discarding $1/\epsilon$. However in our method, to our advantage, there is no need to fix the arbitrary constant, b.

The fermion correlators also provide an opportunity to test whether we are accurately taking into account the vacuum structure of the theory. The correlator $\langle\psi_-(z,\bar{z})\psi_+(0)\rangle$ has its first non-trivial contribution at second order in λ, the first order at which the bubbles contribute. Bosonizing this corrrelator and expanding to second order we obtain

$$\begin{aligned} 2m\sqrt{\frac{\bar{z}}{z}}\mathrm{K}_1(\mathrm{mr}) &= \langle\psi_-(z,\bar{z})\psi_+(0)\rangle \\ &= \langle e^{-i\phi^L(z,\bar{z})}e^{i\phi^L(0)}\rangle \\ &= \frac{1}{z} + \frac{1}{4\pi^2}K(1) + O(\lambda^4), \end{aligned} \tag{5.23}$$

where

$$K(1) = \frac{\lambda^2}{z}\int d^2w_1 d^2w_2 w_1 w_2^{-1}(w_1 - z)^{-1}(w_2 - z)|w_1 - w_2|^{-2}. \qquad (5.24)$$

We note that the zeroth order term of this correlator agrees with expansion of K_1.

The integral K in 5.24 is divergent. (In fact it is quadratically divergent. If the bubbles were explicitly included we would only be facing a logarithmic divergence. That our evaluation of K still leads only to log's indicates we have correctly handled the bubbles.) Regulating $K(1)$ as before by continuing $\hat{\beta}$ we obtain

$$K(\hat{\beta}) = \frac{\lambda^2|z|^{4-2\hat{\beta}^2}}{z^{\hat{\beta}^2}}\int d^2w_1 d^2w_2 w_1^{\hat{\beta}^2} w_2^{-\hat{\beta}^2}(w_1 - 1)^{-\hat{\beta}^2}(w_2 - 1)^{\hat{\beta}^2}|w_1 - w_2|^{-2\hat{\beta}^2}. \qquad (5.25)$$

Using the techniques described in section 4, $K(\hat{\beta})$ is easily evaluated to be

$$\begin{aligned} K(\hat{\beta}) &= -4\lambda^2|z|^{4-2\hat{\beta}^2} z^{-\hat{\beta}^2} \times \qquad (5.26) \\ &\qquad \left[K_1(\hat{\beta})\left(s^2(\hat{\beta}^2)K_2(\hat{\beta}) + \left(s(\hat{\beta}^2)s(2\hat{\beta}^2) + s^2(\hat{\beta}^2)\right)K_2(-\hat{\beta})\right)\right] \end{aligned}$$

where $K_1(\hat{\beta})$ and $K_2(\pm\hat{\beta})$ equal

$$\begin{aligned} K_1(\hat{\beta}) &= \int_0^1 dw_1 \int_0^{w_1} dw_2 (w_1 - w_2)^{-\hat{\beta}^2} = \frac{\mathrm{B}(1-\hat{\beta}^2, 1)}{2-\hat{\beta}^2} \qquad (5.27) \\ K_2(\pm\hat{\beta}) &= \int_0^1 dw_1 w_1^{\hat{\beta}^2-2}(1-w_1)^{\mp\hat{\beta}^2}\int_0^{w_1} dw_2 w_2^{\hat{\beta}^2-2}(1-w_2)^{\pm\hat{\beta}^2}(w_1-w_2)^{-\hat{\beta}^2} \\ &= \hat{\beta}^2(1-\hat{\beta}^2)\Gamma(\hat{\beta}^2-1, 1-\hat{\beta}^2)\mathrm{B}(\hat{\beta}^2-1, 1\mp\hat{\beta}^2) \\ &\qquad {}_3F_2(1\mp\hat{\beta}^2, \hat{\beta}^2, \hat{\beta}^2-1, 2, \hat{\beta}^2\mp\hat{\beta}^2, 1) \\ &= \hat{\beta}^2(1-\hat{\beta}^2)\Gamma^2(\hat{\beta}^2-1)\Gamma(1-\hat{\beta}^2)\Gamma(2-\hat{\beta}^2) \\ &\qquad {}_3F_2(1\pm\hat{\beta}^2, \hat{\beta}^2-1, 2-\hat{\beta}^2, 2, 1, 1), \qquad (5.28) \end{aligned}$$

where in the last line we have used the analytic continuation formula in appendix C. Now writing $\hat{\beta}^2 = 1 + \epsilon$ and taking $\epsilon \to 0$ we find

$$\begin{aligned} K_1(\hat{\beta}) &= -\frac{1}{\epsilon}\left(1 + \epsilon + O(\epsilon^2)\right) \qquad (5.29) \\ K_2(\hat{\beta}) &= \frac{1}{\epsilon^2}\left(\epsilon + O(\epsilon^2)\right) \qquad (5.30) \\ K_2(-\hat{\beta}) &= \frac{1}{\epsilon^2}\left(1 + \epsilon + O(\epsilon^2)\right). \qquad (5.31) \end{aligned}$$

Some of the details of this calculation may be found in appendix D. Combining these results we find $K(1+\epsilon)$ to be

$$\begin{aligned} K(1+\epsilon) &= -4\pi^2|mz|^2|\mu z|^{2-2\hat{\beta}^2} z^{-\hat{\beta}^2}\frac{1}{\epsilon}(1+\epsilon) \\ &= 4\pi^2|mz|^2 z^{-\hat{\beta}^2}\left(-\frac{1}{\epsilon}+2\left(\log(\frac{mr}{2})+\gamma-\frac{1}{2}\right)\right) \end{aligned} \tag{5.32}$$

where we have used $\mu^{-\epsilon} = (1-\epsilon\gamma+O(\epsilon^2))\, m^{-\epsilon}$ and $|z| = r/2$. We again preserve the correct scaling dimensions of $K(\hat{\beta})$ by not expanding $z^{-\hat{\beta}^2}$. Instead we expand only $|z|^{4-2\hat{\beta}^2}$, the z-dependence scaled out from the integral. It is this piece, coming from the integral, that we expect to generate the log's. Dropping the infinite piece in $K(1+\epsilon)$, we find for $\langle\psi_-(z,\bar{z})\psi_+(0)\rangle$

$$\begin{aligned} \langle\psi_-(z,\bar{z})\psi_+(0)\rangle &= \frac{1}{z}+\frac{2}{z}|mz|^2\left(\log(\frac{mr}{2})+\gamma-\frac{1}{2}\right)+O(m^4) \\ &= 2m\sqrt{\frac{\bar{z}}{z}}\left(\frac{1}{mr}+\frac{mr}{2}\left(\log(\frac{mr}{2})+\gamma-\frac{1}{2}\right)\right)+O(m^4). \end{aligned} \tag{5.33}$$

Expanding out $K_1(mr)$, we find the two sides of 5.23 agree. Thus the bubble diagrams, even though not explicitly included, are taken into account.

Having demonstrated that our regularisation techniques work on fermion correlators, we now apply the same methods on the Ising spin correlator with some confidence. We recall the spin correlator is given by

$$\langle\sigma(z,\bar{z})\sigma(0)\rangle^2 = \frac{1}{2}\left(G(\frac{1}{2},-\frac{1}{2})-G(\frac{1}{2},\frac{1}{2})\right). \tag{5.34}$$

From Wu et al. [5], the expansion in the scaling limit of the spin correlator is

$$\langle\sigma(z,\bar{z})\sigma(0)\rangle^2 = \frac{1}{R^{1/2}}\left(1+t\Omega+t^2(\frac{\Omega^2}{4}+\frac{1}{8})+\frac{t^3}{8}\Omega+\cdots\right), \tag{5.35}$$

where $\Omega = \log(\frac{t}{8})+\gamma$, $R \propto r$, and t, the scaling variable, is proportional to mr.

We can use the zeroth order terms on both sides of 5.35 to fix R in terms of r. We find

$$R = 2r \tag{5.36}$$

At the next order, 5.35 reduces to

$$\frac{t\Omega}{R^{1/2}} = \frac{1}{4\pi} L(1), \tag{5.37}$$

were $L(1)$ equals

$$\begin{aligned} L(1) &\equiv -\lambda \int d^2w \langle e^{i\Phi(z)/2} e^{i\Phi(0)} e^{-i\Phi(w)} \rangle \\ &= -\lambda \int d^2w |z|^{1/2} |w-z|^{-1} |w|^{-1}. \end{aligned} \tag{5.38}$$

As with the first order term of the fermion correlator $\langle \bar{\psi}_-(z,\bar{z})\psi_+(0)\rangle$, this integral has a logarithmic divergence. Regulating L as before by taking $L(1) \to L(\hat{\beta})$, we find

$$L(\hat{\beta}) = -\lambda |z|^{\hat{\beta}^2/2} |z|^{2-2\hat{\beta}^2} \int d^2w |w-1|^{-\hat{\beta}^2} |w|^{-\hat{\beta}^2}. \tag{5.39}$$

This integral has been done previously (see 4.27) with the result

$$L(\hat{\beta}) = 2\lambda |z|^{\hat{\beta}^2/2} |z|^{2-2\hat{\beta}^2} s(\hat{\beta}^2/2) \mathrm{B}(1-\hat{\beta}^2/2, 1-\hat{\beta}^2/2) \mathrm{B}(1-\hat{\beta}^2/2, \hat{\beta}^2-1). \tag{5.40}$$

Taking $\hat{\beta}^2$ to 1 via $\hat{\beta}^2 = 1+\epsilon$ we find

$$L(1+\epsilon) = -2\pi m |z|^{\hat{\beta}^2/2} \mu^{\hat{\beta}^2-1} |\mu z|^{2-2\hat{\beta}^2} \left(\frac{1}{\epsilon} + 4\log(2)\right). \tag{5.41}$$

As before we do not expand $|z|^{\hat{\beta}^2/2}\mu^{\hat{\beta}^2-1}$ so as to preserve the correct scaling dimension. We then have

$$L(1) = 4\pi m |z|^{1/2} (\log(\frac{mr}{8}) + \gamma). \tag{5.42}$$

Taking the scaling variable, t, to equal mr we see $L(1) = \frac{4\pi t}{R^{1/2}}\Omega$. Comparing this with 5.35 we see we have agreement.

At second order, the perturbative expansion of the spin correlator gives

$$\frac{1}{R^{1/2}} t^2 \left(\frac{\Omega^2}{4} + \frac{1}{8}\right) = \frac{1}{8\pi^2} M(1) \tag{5.43}$$

where $M(1)$ equals

$$\begin{aligned} M(1) &= \lambda^2 \int d^2w d^2y \langle e^{i\Phi(z)/2} e^{-i\Phi(0)/2} e^{i\Phi(w)} e^{i\Phi(y)} \rangle \\ &= \lambda^2 |z|^{3/2} \int d^2w d^2y |w-y|^{-2} |w-1| |w|^{-1} |y-1|^{-1} |y|. \quad (5.44) \end{aligned}$$

We will show 5.43 is valid to the leading log term.

$M(1)$ is (quadratically) divergent. Again, as with the second order term of the fermion correlator, explicit inclusion of the bubble contribution leaves $M(1)$ logarithmically divergent. But, as before, explicit inclusion of the bubbles is unnecessary. To regulate, we continue $\hat{\beta}$. $M(\hat{\beta})$ is then

$$M(\hat{\beta}) = \lambda^2 |z|^{4-5\hat{\beta}^2/2} \int d^2w d^2y |w-y|^{-2\hat{\beta}^2} |w-1|^{\hat{\beta}^2} |w|^{-\hat{\beta}^2} |y-1|^{-\hat{\beta}^2} |y|^{\hat{\beta}^2}. \quad (5.45)$$

We have already evaluated this integral. Copying the result in 4.18 gives us

$$\begin{aligned} M(\hat{\beta}) &= -4\lambda^2 |z|^{4-5\hat{\beta}^2/2} \Big[J_+ \Big(s^2(\hat{\beta}^2/2) K_- + s(\hat{\beta}^2/2) s(3\hat{\beta}^2/2) K_+ \Big) \\ &\qquad + J_- \Big(s^2(\hat{\beta}^2/2) K_+ - s^2(\hat{\beta}^2/2) K_- \Big) \Big]. \quad (5.46) \end{aligned}$$

where

$$\begin{aligned} K_\pm(\hat{\beta}) &= \pm \mathrm{B}(1 \pm \hat{\beta}^2/2, \hat{\beta}^2 - 1) \Gamma(1-\hat{\beta}^2) \Gamma(\hat{\beta}^2 - 1) (\frac{\hat{\beta}^2}{2}) (\hat{\beta}^2 - 1) \times \quad (5.47) \\ &\quad {}_3F_2(1 \pm \hat{\beta}^2/2, \hat{\beta}^2, \hat{\beta}^2 - 1, \hat{\beta}^2 \pm \hat{\beta}^2/2, 2, 1) \\ J_\pm(\hat{\beta}) &= \mathrm{B}(1 \pm \hat{\beta}^2/2, 2 - \hat{\beta}^2) \mathrm{B}(1 - \hat{\beta}^2, 1 \pm \hat{\beta}^2/2) \times \quad (5.48) \\ &\quad {}_3F_2(\pm\hat{\beta}^2/2, 2 - \hat{\beta}^2, 1 \pm \hat{\beta}^2/2, 3 - \hat{\beta}^2 \pm \hat{\beta}^2/2, 2 - \hat{\beta}^2 \pm \hat{\beta}^2/2, 1). \end{aligned}$$

Setting $\hat{\beta}^2 = 1 + \epsilon$ we find

$$\begin{aligned} K_\pm(\epsilon) &= \mp \frac{1}{2\epsilon^2} \Big[1 + \epsilon(2 - \gamma - \psi(1 \pm 1/2)) + O(\epsilon^2) \Big] \quad (5.49) \\ J_\pm(\epsilon) &= -\frac{1}{\epsilon} \Big[1 + \epsilon(\psi(1 \pm 1/2) + \gamma \mp 1 \pm \pi^2/4) + O(\epsilon^2) \Big] \quad (5.50) \end{aligned}$$

where ψ is the logarithmic derivative of the gamma-function, i.e. $\psi(x) = \Gamma'(x)/\Gamma(x)$. The details of this calculation may be found in appendix E.

Putting everything together we find for $M(1+\epsilon)$

$$\begin{aligned} M(1+\epsilon) &= -4|z|^{4-5\hat{\beta}^2/2}\lambda^2\frac{1}{\epsilon^2}\left(-\frac{\pi^2}{2}+O(\epsilon^2)\right) \\ &= \frac{2\pi^2}{\epsilon^2}|z|^{-\hat{\beta}^2/2}|zm|^2\left(1-2\epsilon\log(z\mu)+2\epsilon^2\log^2(z\mu)\right) \end{aligned} \tag{5.51}$$

where in the last line we have substituted $-m\mu^{1-\hat{\beta}^2}$ for λ. The presence of the $1/\epsilon^2$ term indicates, as expected, that the leading log term is $\log^2$. Substituting in t and $R^{1/2}$, subtracting the infinite terms, and dropping all but the $\log^2$ terms, we find

$$M(1) = \frac{t^2}{R^{1/2}}\left(2\pi^2\Omega^2 + O(\Omega)\right). \tag{5.52}$$

Comparing with 5.35 we find the coefficients of the leading log's at second order agree.

The ability to reproduce the non-trivial behavior of the Ising spin correlators provides us with a degree of confidence that our expression for the correlators away from $\hat{\beta}=1$ are correct.

5.2 Gross-Neveu SU(2)

We now go on to demonstrate that our methods of evaluating the sine-Gordon correlators reproduce the known behaviour of the Gross-Neveu model (sine-Gordon at $\hat{\beta}^2=2$). Specifically we demonstrate that we are able to reproduce the known β-functions for λ and $\hat{\beta}$ governing the Kosterlitz-Thouless transition to lowest order, as say calculated by Amit et al. [12] and Boyanovsky [13].

As shown in section 3, the sine-Gordon theory develops UV singularities as $\hat{\beta}^2 \to 2$. As with Ising at $\hat{\beta}=1$, these singularities manifest themselves as poles in gamma functions. And similiarly to Ising, we regulate the correlators by continuing away from $\hat{\beta}^2=2$ via $\hat{\beta}^2=2-\epsilon$. However unlike Ising, we handle these singularites, because they are UV, with a conventional renormalisation.

In order to facilitate comparison with Amit et al. [12] and Boyanovsky [13] we employ a sine-Gordon action equivalent to theirs:

$$S = -\frac{1}{4\pi}\int d^2z\left[\partial_z\Phi\partial_{\bar{z}}\Phi + \frac{2\lambda}{\hat{\beta}^2}cos(\hat{\beta}\Phi)\right], \tag{5.1}$$

where the free propagator is

$$\langle\Phi(z)\Phi(0)\rangle = -\log(4z\bar{z}). \tag{5.2}$$

The difference in the propagators amounts to a difference in how its IR singularities are cured.

To renormalise the theory we employ the following renormalisation prescription

$$\begin{aligned}
\Phi_0^2 &= Z_\Phi \Phi_R^2; \\
\hat{\beta}_0^2 &= Z_\Phi^{-1}\hat{\beta}_R^2; \\
(\cos(\hat{\beta}\Phi))_0 &= Z_C(\cos(\hat{\beta}\Phi))_R; \\
\lambda_0 &= Z_C^{-1} Z_\lambda Z_\Phi^{-1}\lambda_R;
\end{aligned} \tag{5.3}$$

where the bare quantities are listed on the left hand side and the renormalized quantities on the right hand side. λ_R is understood to be dimensionless. This prescription, like Amit et al.'s and Boyanovsky's, involves a trivial renormalisation of the $\hat{\beta}$ so as to ensure $\hat{\beta}_0\Phi_0 = \hat{\beta}_R\Phi_R$. However it differs from their's in that it introduces a wavefunction renormalisation of the $\cos(\hat{\beta}\Phi)$. That this renormalisation is necessary is evident from considering the correlator $\langle\cos(\hat{\beta}\Phi(z))\cos(\hat{\beta}\Phi(0))\rangle$. It possesses singularities at $O(\lambda^2)$, the only way which to remove is through a wavefunction renormalisation. A consequence of this is that the β-function for λ has been computed incorrectly by both these authors. (We note that Amit et al.'s [12] calculation of β_λ at higher orders was already known to be flawed because of his method of regulating the theory. See C. Lovelace [14] for a detailed discussion.) However as we are interested in reproducing the β-functions only at lowest order as a check on our methodology, we will not concern ourselves here with correcting this error.

The wavefunction renormalisation for Φ, Z_Φ, is found through calculating the correlator

$$\langle\partial_z\Phi(z)\partial_z\Phi(0)\rangle. \tag{5.4}$$

Because this correlator is even in λ, Z_Φ takes the form $Z_\Phi = \sum_{n=0}^{\infty} a_{2n}\lambda^{2n}$. Wavefunction renormalisation of $\cos(\hat{\beta}\Phi)$ may be computed through the correlator $\langle\cos(\hat{\beta}\Phi(z))\cos(\hat{\beta}\Phi(0))\rangle$. Knowing the wavefunction renormalisations of $\cos(\hat{\beta}\Phi)$ and of Φ allows the computation of the coupling constant renormalisation, Z_λ, through the correlator $\langle\partial_z\Phi\cos(\hat{\beta}\Phi)\rangle$. Because $\langle\partial_z\Phi\cos(\hat{\beta}\Phi)\rangle$

only has terms odd in λ (and is finite at $O(\lambda)$), Z_λ takes the form

$$Z_\lambda = \mu^{-\epsilon}(1 + \sum_{n=1}^{\infty} a_{2n}\lambda^{2n}). \tag{5.5}$$

μ is an arbitrary mass parameter introduced to make the renormalised coupling dimensionless. Here we will not be interested in computing the higher order terms. So we have (rather trivially)

$$Z_\lambda = \mu^{-\epsilon}(1 + O(\lambda^2)), \tag{5.6}$$

and we are left to compute Z_Φ. (We do not need to worry over Z_C as it contributes to the renormalisation of λ at $O(\lambda^2)$, and so makes a contribution to λ's β-function only at $O(\lambda^3)$.)

To compute Z_Φ, we employ a trick to calculate $\langle \partial_z \Phi(z) \partial_z \Phi(0)\rangle$. We can write

$$\langle \partial_z \Phi(z) \partial_z \Phi(0)\rangle = \lim_{\alpha \to 0} \frac{1}{\alpha^2} \partial_x \partial_y \langle e^{i\alpha\Phi(x)} e^{-i\alpha\Phi(y)}\rangle|_{\substack{x=z \\ y=0}}. \tag{5.7}$$

Thus we can use our results from Section 4. Doing so and taking the $\alpha \to 0$ limit we find

$$\begin{aligned} \langle \partial_z \Phi_0(z) \partial_z \Phi_0(0)\rangle \;\; &= \;\; -(z)^{-2} - \\ & \frac{\lambda_0^2}{16 \cdot 2^{2\hat{\beta}_R^2} \pi^2 \hat{\beta}_R^4} (2 - \hat{\beta}_R^2)(1 - \hat{\beta}_R^2)(z)^{-\hat{\beta}_R^2} (\bar{z})^{2-\hat{\beta}_R^2} L, \end{aligned} \tag{5.8}$$

where L is

$$L = -8\pi \hat{\beta}_R^2 s(\hat{\beta}_R^2)(\hat{\beta}_R^2 - 1) \frac{\Gamma^2(1 - \hat{\beta}_R^2)\Gamma(\hat{\beta}_R^2 - 1)\Gamma(2 - \hat{\beta}_R^2)}{\Gamma^2(3 - \hat{\beta}_R^2)}. \tag{5.9}$$

Writing $\hat{\beta}_R^2 = 2 - \epsilon$, the singularities in the correlator become

$$\langle \partial_z \Phi(z) \partial_z \Phi(0)\rangle = -(z)^{-2} + \frac{\lambda^2}{64\epsilon}(z)^{-\hat{\beta}_R^2} (\bar{z})^{2-\hat{\beta}_R^2} \left(1 + \epsilon(1/2 + 2\log(2))\right). \tag{5.10}$$

Using minimal subtraction, we find Z_Φ to be

$$Z_\Phi = 1 - \frac{\mu^{-2\epsilon}\lambda_0^2}{64\epsilon} + O(\lambda^4). \tag{5.11}$$

Knowing Z_Φ and Z_λ allows us to calculate the β-functions for β and λ:

$$\beta_\lambda = \mu\frac{\partial\lambda_R}{\partial\mu} = \mu\lambda_0\frac{\partial\mu^{-\epsilon}}{\partial\mu} = -\epsilon\lambda_R + O(\lambda^3); \tag{5.12}$$

$$\beta_{\hat\beta} = \mu\frac{\partial\hat\beta_R}{\partial\mu} = \mu\hat\beta_0\frac{\partial Z_\Phi^{1/2}}{\partial\mu} = \frac{\hat\beta_R\lambda_R^2}{64} + O(\lambda^4). \tag{5.13}$$

Defining $\delta = -\epsilon/2$, we can recast the above in a form directly comparable with Amit et al.'s [12] and Boyanovsky's [13] result

$$\beta_\lambda = 2\delta\lambda_R + O(\lambda^3); \tag{5.14}$$

$$\beta_\delta = \hat\beta_R\beta_{\hat\beta} = \frac{\hat\beta_R^2\lambda_R^2}{64} + O(\lambda^4) = \frac{\lambda_R^2}{32} + \frac{\delta\lambda_R^2}{32} + O(\lambda^4). \tag{5.15}$$

The leading terms of these β-functions agree with their previous results. Thus we again have shown that this methodology is consistent with other methods.

6 Discussion

We have shown how the short distance expansion of some sine-Gordon correlation functions is well defined and reasonably tractable. The ultimate goal is to find some non-perturbative characterization of these correlation functions. We make two remarks. First of all, unlike the large distance expansion, the short distance expansions at and away from the free fermion point are of comparable complexity. Secondly, we have in no way utilized the integrability structure of the sine-Gordon theory, i.e. factorizable S-matrix, quantum inverse scattering method, etc. in developing the short distance expansion. It would be very interesting to understand the consequences of integrability in this context.

The results presented herein leave two concrete avenues open for further exploration. The first arises from our knowledge of the differential equation the correlators of the vertex operators satisfy at the free fermion point. Knowing the first set of terms in the perturbative expansion should allow, through the differential equation, the generation of higher order terms in the expansion.

The second is a correct evaluation of the β-function at $O(\lambda^3)$ for the coupling λ in the SU(2) Gross-Neveu model. To make this evaluation two calculations would be necessary. The wavefunction renormalisation of $\cos(\hat{\beta}\Phi)$ would need to be calculated through the correlator $\langle\cos(\hat{\beta}\Phi(z))\cos(\hat{\beta}\Phi(0))\rangle$. Knowing this then allows the calculation of Z_λ, a piece of the coupling constant renormalisation, at $O(\lambda^2)$ through the correlator $\langle\partial_z\Phi(z)\cos(\hat{\beta}\Phi(0))\rangle$. From there, β_λ is directly calculable.

7 Acknowledgements

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A A Generalized Euler Integral

In this appendix we evaluate the following generalized Euler integral,

$$
\begin{aligned}
I \;=\; & \int_0^1 dw_1 w_1^{a_1}(1-w_1)^{b_1} \times \\
& \int_0^{w_1} dw_2 w_2^{a_2}(1-w_2)^{b_2}(w_1-w_2)^{\alpha_{12}} \times \\
& \int_0^{w_2} dw_3 w_3^{a_3}(1-w_3)^{b_3}(w_1-w_3)^{\alpha_{13}}(w_2-w_3)^{\alpha_{23}} \times \\
& \qquad\qquad\qquad \vdots \\
& \int_0^{w_{n-1}} dw_n w_n^{a_n}(1-w_n)^{b_n}(w_1-w_n)^{\alpha_{1n}}\cdots(w_{n-1}-w_n)^{\alpha_{n-1,n}}.
\end{aligned} \tag{A.1}
$$

Employing the following series of changes of variables,

$$
w_2 = w_2' w_1,\;\; w_3 = w_3' w_2' w_1,\;\; \ldots\;,\;\; w_n = w_n' \cdots w_2' w_1,
$$

the above integral reduces to

$$
\begin{aligned}
I \;=\; & \int_0^1 dw_1 w_1^{\sum_{i=1}^n a_i + \sum_{i<j}^n \alpha_{ij} + n-1}(1-w_1)^{b_1} \times \\
& \int_0^1 dw_2 w_2^{\sum_{i=2}^n a_i + \sum_{1<i<j}^n \alpha_{ij} + n-2}(1-w_1w_2)^{b_2}(1-w_2)^{\alpha_{12}} \times \\
& \int_0^1 dw_3 w_3^{\sum_{i=3}^n a_i + \sum_{2<i<j}^n \alpha_{ij} + n-3}(1-w_1w_2w_3)^{b_3}(1-w_2w_3)^{\alpha_{13}} \times \\
& \qquad\qquad\qquad\qquad\qquad\qquad (1-w_3)^{\alpha_{23}} \times \\
& \qquad\qquad\qquad \vdots \\
& \int_0^1 dw_n w_n^{a_n}(1-w_1\cdots w_n)^{b_n}(1-w_2\cdots w_n)^{\alpha_{1n}}(1-w_3\cdots w_n)^{\alpha_{2n}}\cdots \\
& \qquad\qquad\qquad\qquad (1-w_{n-1}w_n)^{\alpha_{n-2,n}}(1-w_n)^{\alpha_{n-1,n}}.
\end{aligned} \tag{A.2}
$$

Using the expansion,

$$
(1-w)^a = \sum_{k=0}^{\infty} \frac{(-a)_k}{k!} w^k, \tag{A.3}
$$

where $(x)_k \equiv x(x+1)\cdots(x+k-1) \equiv \Gamma(x+k)/\Gamma(x)$, the above becomes

$$
\begin{aligned}
I \;=\; & \int_0^1 dw_1 w_1^{\sum_{i=1}^n a_i + \sum_{i<j}^n \alpha_{ij} + n-1} (1-w_1)^{b_1} \times && \text{(A.4)} \\
& \sum_{k_{12}} \frac{(-b_2)_{k_{12}}}{k_{12}!} \int_0^1 dw_2 w_2^{\sum_{i=2}^n a_i + \sum_{1<i<j}^n \alpha_{ij} + n-2} (1-w_2)^{\alpha_{12}} (w_1 w_2)^{k_{12}} \times \\
& \sum_{\substack{k_{23} \\ k_{123}}} \frac{(-b_3)_{k_{123}} (-\alpha_{13})_{k_{23}}}{k_{123}! k_{23}!} \int_0^1 dw_3 w_3^{\sum_{i=3}^n a_i + \sum_{2<i<j}^n \alpha_{ij} + n-3} (1-w_3)^{\alpha_{23}} \times \\
& \qquad (w_1 w_2 w_3)^{k_{123}} (w_2 w_3)^{k_{23}} \times \\
& \qquad \vdots \\
& \sum_{\substack{k_{1\cdots n} \\ k_{2\cdots n} \\ \vdots \\ k_{n,n-1}}} \frac{(-b_n)_{k_{1\cdots n}} (-\alpha_{1n})_{k_{2\cdots n}} (-\alpha_{2n})_{k_{3\cdots n}} \cdots (-\alpha_{n-2,n})_{k_{n-1,n}}}{k_{1\cdots n}! k_{2\cdots n}! \cdots k_{n,n-1}!} \\
& \times \int_0^1 dw_n w_n^{a_n} (1-w_n)^{\alpha_{n,n-1}} (w_1 \cdots w_n)^{k_{1\cdots n}} \cdots (w_n w_{n-1})^{k_{n,n-1}}.
\end{aligned}
$$

where the $k_{m\cdots n}$ arise from the expansion of $(1 - w_m \cdots w_n)$. Defining the following notation

$$
\sigma_m = \sum_{i=m}^n a_i + \sum_{i \neq j > m-1}^n \alpha_{ij} + (n-m), \tag{A.5}
$$

$$
\begin{aligned}
\sum_m k \;=\; & \text{sum of k's with index m} && \text{(A.6)} \\
=\; & k_{1\cdots m} + k_{1\cdots m,m+1} + \cdots + k_{1\cdots m\cdots n} + \\
& k_{2\cdots m} + k_{2\cdots m,m+1} + \cdots + k_{2\cdots m\cdots n} + \\
& \vdots \\
& k_{m,m+1} + k_{m,m+1,m+2} + \cdots + k_{m\cdots n},
\end{aligned}
$$

and

$$
(k)! = k_{12}! k_{123}! \cdots k_{1\cdots n}! \times k_{23}! \cdots k_{2\cdots n}! \times \cdots \times k_{n,n-1}!, \tag{A.7}
$$

I simplifies to

$$
\begin{aligned}
I &= \sum_{\{k\}} \prod_{i=2}^{n} (-b_i)_{k_{1\cdots i}} \prod_{i<j-1}^{n} (-\alpha_{ij})_{k_{i+1\cdots j}}/(k)! \times \qquad \text{(A.8)}\\
&\int_0^1 dw_1 w_1^{\sigma_1+\sum_1 k}(1-w_1)^{b_1} \times \\
&\int_0^1 dw_2 w_2^{\sigma_2+\sum_2 k}(1-w_1)^{\alpha_{12}} \times \\
&\qquad\vdots \\
&\int_0^1 dw_n w_n^{\sigma_n+\sum_n k}(1-w_1)^{\alpha_{n-1,n}}
\end{aligned}
$$

Doing these integrals, we are able to put I into its final form:

$$
\begin{aligned}
I &= \mathrm{B}(b_1+1,\sigma_1+1)\mathrm{B}(\alpha_{12}+1,\sigma_2+1)\cdots B(\alpha_{n-1,n}+1,\sigma_n+1) \times \\
&\sum_{\{k\}} \prod_{i=2}^{n} (-b_i)_{k_{1\cdots i}} \prod_{i<j-1}^{n} (-\alpha_{ij})_{k_{i+1\cdots j}}/(k)! \times \\
&(\sigma_1+1)_{\sum_1 k}/(\sigma_1+b_1+2)_{\sum_1 k} \times \qquad \text{(A.9)}\\
&(\sigma_2+1)_{\sum_2 k}/(\sigma_1+\alpha_{12}+2)_{\sum_2 k} \times \\
&\qquad\vdots \\
&(\sigma_n+1)_{\sum_n k}/(\sigma_n+\alpha_{n-1,n}+2)_{\sum_2 k}.
\end{aligned}
$$

B Evaluation of K(i,j)

In this appendix we evaluate the integrals $K(i,j)$ introduced in Section 4.2. Using Appendix A we can write $K(i,j)$ as

$$
\begin{aligned}
K(i,j) &= \int_0^1 dw_i w_i^{-2+\hat\beta^2}(1-w_i)^{\gamma_i} \int_0^{w_i} dw_j w_j^{-2+\hat\beta^2}(1-w_j)^{\gamma_j}(w_i-w_j)^{-\hat\beta^2} \\
&= \mathrm{B}(\gamma_i+1,\hat\beta^2-2)\mathrm{B}(1-\hat\beta^2,\hat\beta^2-1) \times \qquad \text{(B.1)}\\
&\sum_{k=0}^{\infty} \frac{(-\gamma_j)_k(\hat\beta^2-2)_k(\hat\beta^2-1)_k}{k!(\gamma_i+\hat\beta^2-1)_k(0)_k}
\end{aligned}
$$

where $\mathrm{B}(x,y)=\Gamma(x)\Gamma(y)/\Gamma(x+y)$, $(a)_k=\Gamma(a+k)/\Gamma(a)$, and the sum $\sum_k$ is the standard hypergeometric function,

$$
{}_3F_2(-\gamma_j,\hat\beta^2-2,\hat\beta^2-1,\gamma_i+\hat\beta^2-1,0,1). \qquad \text{(B.2)}
$$

However this form is somewhat problematic as a pole in ${}_3F_2$ is multiplying a zero in B. To make sense of this expression we continue $w_j^{-2+\hat{\beta}^2}$ to $w_j^{-2+\hat{\beta}^2+\epsilon}$ in the above integrand and take the limit $\epsilon \to 0$. With this continuation, $K(i,j)$ equals

$$K(i,j) = \mathrm{B}(\gamma_i + 1, \hat{\beta}^2 - 2)\mathrm{B}(1 - \hat{\beta}^2, \hat{\beta}^2 - 1 + \epsilon) \times \tag{B.3}$$
$$ {}_3F_2(-\gamma_j, \hat{\beta}^2 - 2, \hat{\beta}^2 - 1, \gamma_i + \hat{\beta}^2 - 1, \epsilon, 1). \tag{B.4}$$

Using the identity

$$\begin{aligned} &\lim_{\zeta \to -n} \frac{{}_3F_2(\alpha, \beta, \gamma, \delta, \zeta, z)}{\Gamma(\zeta)} = \\ &\quad \frac{\alpha(\alpha+1)\cdots(\alpha+n)\beta(\beta+1)\cdots(\beta+n)\gamma(\gamma+1)\cdots(\gamma+n)}{(n+1)!\delta(\delta+1)\cdots(\delta+n)} \times \\ &\quad z^{n+1}{}_3F_2(\alpha+n+1, \beta+n+1, \gamma+n+1, \delta+n+1, 2, z), \end{aligned} \tag{B.5}$$

and then taking the limit $\epsilon \to 0$, we obtain

$$\begin{aligned} K(i,j) = B(1+\gamma_i, \hat{\beta}^2 - 1)\Gamma(1-\hat{\beta}^2)\Gamma(\hat{\beta}^2 - 1)(-\gamma_j)(\hat{\beta}^2 - 1) \times \\ {}_3F_2(1-\gamma_j, \hat{\beta}^2 - 1, \hat{\beta}^2, \hat{\beta}^2 + \gamma_i, 2, 1), \end{aligned} \tag{B.6}$$

where we have also used the relation $x\Gamma(x) = \Gamma(x+1)$.

C Analytic Continuation of ${}_3F_2$

The Gaussian hypergeometric function ${}_3F_2(\alpha, \beta, \gamma, \delta, \zeta, 1)$ is, in general, a meromorphic function of the variables $\alpha, \beta, \gamma, \delta$, and ζ. However its sum representation (as say given in Appendix B) is not necessarily convergent for all values of these variables. The convergence of the sum is determined by defining the "excess" parameter, s, by

$$s = \delta + \zeta - (\alpha + \beta + \gamma). \tag{C.1}$$

If $s > 0$ the sum converges. However we will often encounter situations where $s < 0$ and the sum representation is no good. To get around this problem analytic continutation formulae are needed. The only one that will prove to be necessary is an identity known as Dixon's theorem:

$$
{}_3F_2(\alpha,\beta,\gamma,\delta,\zeta,1) = \frac{\Gamma(\delta)\Gamma(\zeta)\Gamma(s)}{\Gamma(\alpha)\Gamma(s+\beta)\Gamma(s+\gamma)} {}_3F_2(\delta-\alpha,\zeta-\alpha,s,s+\beta,s+\gamma,1). \tag{C.2}
$$

The convergence of the hypergeometric function of the right hand side is then determined by the positivity of α. Notice that because ${}_3F_2$ is invariant under permutations of α, β, and γ and permutations of δ and ζ, variations (144 in total - not all necessarily distinct) of this identity exist.

D Taking $\hat{\beta} \to 1$ in $\langle \psi_-(z,\bar{z})\psi_+(0)\rangle$

We focus on the Taylor series expansions of the hypergeometric functions in $K_2(\pm\hat{\beta})$. The expansion of the ${}_3F_2$ function in $K_2(\hat{\beta})$ is given by

$$
\begin{aligned}
{}_3F_2(1+\hat{\beta}^2,\hat{\beta}^2-1,2-\hat{\beta}^2,2,1,1) \quad &= \quad \frac{\Gamma(1-\hat{\beta}^2)(1-\hat{\beta}^2)^2(-\hat{\beta}^2)}{\Gamma(1+\hat{\beta}^2)\Gamma(3-2\hat{\beta}^2)(3-2\hat{\beta}^2)} \times \\
&\quad {}_3F_2(2-\hat{\beta}^2,1-\hat{\beta}^2,2-\hat{\beta}^2,2,4-2\hat{\beta}^2,1),
\end{aligned} \tag{D.1}
$$

where we have used Dixon's theorem, C.2, and the identity, B.2. Evaluating further we have

$$
\begin{aligned}
{}_3F_2(1+\hat{\beta}^2,\hat{\beta}^2-1,2-\hat{\beta}^2,2,1,1) \quad &= \quad \frac{\Gamma(2-\hat{\beta}^2)(1-\hat{\beta}^2)(-\hat{\beta}^2)}{\Gamma(1+\hat{\beta}^2)\Gamma(3-2\hat{\beta}^2)(3-2\hat{\beta}^2)} \times \\
&\quad {}_3F_2(2-\hat{\beta}^2,1-\hat{\beta}^2,2-\hat{\beta}^2,2,4-2\hat{\beta}^2,1) \qquad \text{(D.2)} \\
&= \quad \epsilon + O(\epsilon^2).
\end{aligned}
$$

The ${}_3F_2$ function in $K_2(-\hat{\beta})$ is simpler to evaluate:

$$
\begin{aligned}
{}_3F_2(1-\hat{\beta}^2,\hat{\beta}^2-1,2-\hat{\beta}^2,2,1,1) \quad &= \quad {}_3F_2(-\epsilon,\epsilon,1+\epsilon,2,1,1) \\
&= \quad 1+O(\epsilon^2).
\end{aligned} \tag{D.3}
$$

E Taking $\hat{\beta} \to 1$ in $\langle \sigma(z,\bar{z})\sigma(0)\rangle^2$

As in appendix D, we focus only on the hypergeometric functions that appear in the expressions that are being expanded about $\hat{\beta} = 1$. Appearing in $K_\pm(\hat{\beta})$

we have the function:

$$S_{K_\pm}(\hat{\beta}) = {}_3F_2(1 \pm \hat{\beta}^2/2, \hat{\beta}^2, \hat{\beta}^2 - 1, \hat{\beta}^2 \pm \hat{\beta}^2/2, 2, 1). \tag{E.1}$$

To $O(\epsilon)$, $S_{K_\pm}(\hat{\beta})$ is

$$\begin{aligned} S_{K_\pm}(\hat{\beta}) &= {}_3F_2(1 \pm 1/2, 1, \epsilon, 1 \pm 1/2, 2, 1) + O(\epsilon^2) \\ &= \sum_{k=0}^{\infty} \frac{(\epsilon)_k (1)_k}{(2)_k k!} + O(\epsilon^2) \\ &= \frac{\Gamma(2)\Gamma(1-\epsilon)}{\Gamma(2-\epsilon)\Gamma(1)} + O(\epsilon^2) \\ &= 1 + \epsilon + O(\epsilon^2), \end{aligned} \tag{E.2}$$

where we have used the Gaussian summation formula

$${}_2F_1(a, b, c, 1) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{E.3}$$

In $J_\pm$ we face the sum

$$S_{J_\pm}(\hat{\beta}) = {}_3F_2(\pm\hat{\beta}^2/2, 2-\hat{\beta}^2, 1 \pm \hat{\beta}^2/2, 3 - \hat{\beta}^2 \pm \hat{\beta}^2/2, 2 - \hat{\beta}^2 \pm \hat{\beta}^2/2, 1). \tag{E.4}$$

Using the analytic continuation formula in appendix C, $S_{J_\pm}(\hat{\beta})$ can be transformed into

$$S_{J_\pm}(\hat{\beta}) = \frac{\Gamma(3 - \hat{\beta}^2 \pm \hat{\beta}^2/2)\Gamma(2 - \hat{\beta}^2)}{\Gamma(1 \pm \hat{\beta}^2/2)\Gamma(4 - 2\hat{\beta}^2)} \bar{S}_{J_\pm}(\hat{\beta}) \tag{E.5}$$

where $\bar{S}_{J_\pm}(\hat{\beta})$ is

$$\bar{S}_{J_\pm}(\hat{\beta}) = {}_3F_2(1 - \hat{\beta}^2, 2 - \hat{\beta}^2, 2 - \hat{\beta}^2, 2 - \hat{\beta}^2 \pm \hat{\beta}^2/2, 4 - 2\hat{\beta}^2, 1). \tag{E.6}$$

To $O(\epsilon)$, $\bar{S}_{J_\pm}(\hat{\beta})$ is

$$\begin{aligned} \bar{S}_{J_\pm}(\epsilon) &= {}_3F_2(-\epsilon, 1, 1, 1 \pm 1/2, 2, 1) + O(\epsilon^2) \\ &= 1 - \epsilon \sum_{k=1}^{\infty} \frac{(1)_{k-1}(1)_k (1)_k}{k!(1 \pm 1/2)_k (2)_k}. \end{aligned} \tag{E.7}$$

The evaluations of the two sums in E.7 are simlar. So we will only be explicit in the evaluation of $\bar{S}_{J_+}(\epsilon)$.

We may write

$$\begin{aligned}\bar{S}_{J_+}(\epsilon) &= 1 - \epsilon \sum_{k=1}^{\infty} \frac{(1)_{k-1}(1)_k}{(3/2)_k(2)_k} \\ &= 1 - \epsilon \lim_{\delta\to 0} \sum_{k=2}^{\infty} \frac{(1+\delta)_{k-2}(1+\delta)_{k-1}}{(3/2)_{k-1}(2)_{k-1}} \\ &= 1 - \frac{\epsilon}{2} \lim_{\delta\to 0} \frac{\Gamma(\delta-1)\Gamma(\delta)}{\Gamma^2(1+\delta)} \sum_{k=2}^{\infty} \frac{(\delta-1)_k(\delta)_k}{(1/2)_k k!}.\end{aligned} \tag{E.8}$$

Now using the Gaussian summation formula we find

$$\bar{S}_{J_+}(\epsilon) = 1 - \frac{\epsilon}{2} \lim_{\delta\to 0} \frac{\Gamma(\delta-1)\Gamma(\delta)}{\Gamma^2(1+\delta)} \left[\frac{\Gamma(3/2-2\delta)\Gamma(1/2)}{\Gamma(3/2-\delta)\Gamma(1/2-\delta)} - 1 - 2\delta(\delta-1) \right] \tag{E.9}$$

Evaluating this limit we obtain

$$\bar{S}_{J_+}(\epsilon) = 1 + \epsilon(\frac{\pi^2}{4} - 3) + O(\epsilon^2). \tag{E.10}$$

Similarly we find for $\bar{S}_{J_-}(\epsilon)$

$$\bar{S}_{J_-}(\epsilon) = 1 - \epsilon(\frac{\pi^2}{2} + 2) + O(\epsilon^2). \tag{E.11}$$

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